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Numerical Iterative Methods for Solving Nonlinear Volterra-Fredholm Integral Equations

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Abstract

In this paper, the Adomian Decomposition Method (ADM) and Variational Iteration Method (VIM) are used to solve the nonlinear Volterra-Fredholm integral equations. We described the methods, used them on one test problem, and compared the results with their exact solutions in order to demonstrate the validity and applicability of the methods. Moreover, we studied some new uniqueness and convergence results of the solutions.

Keywords : Adomian decomposition method; Variational iteration method; Volterra-Fredholm integral equation; Uniqueness and convergence.

Mathematics Subject Classification (2010): 49M27, 65K10, 65R20, 41A52.

1. INTRODUCTION

In this paper, we consider the nonlinear Volterra-Fredholm integral equations of the form:

$$u(x) = f(x) + (KQu)(x),\ x \in I = [a, b], \tag{1}$$

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with

$$(KQu)(x) = \lambda_1 \int_a^x K_1(x,t)g_1(t,u(t))dt + \lambda_2 \int_a^b K_2(x,t)g_2(t,u(t))dt, \quad (2)$$

where $a, b, \lambda_1, \lambda_2$ are constant values and $\lambda_1, \lambda_2 \neq 0$, also $f(x), K_1(x,t)$, $K_2(x,t), g_1(t,u(t))$ and $g_2(t,u(t))$ are functions that have suitable derivatives on an interval $a \leq t \leq x \leq b$, and $u(t)$ is unknown function. If we set $g_1(t,u(t)) = G_1(u(t))$, $g_2(t,u(t)) = G_2(u(t))$, where G_1 and G_2 are known smooth functions nonlinear in $u(t)$, and where $u(t)$ is an unknown function. We can rewrite Eqs. (1) and (2) as:

$$u(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)G_1(u(t))dt + \lambda_2 \int_a^b K_2(x,t)G_2(u(t))dt. \quad (3)$$

The nonlinear Volterra-Fredholm integral equations (3) arises in a variety of applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, antenna synthesis problem, communicational theory, mathematical economics, population genetics, radiation, the particle transport problems of astrophysics and reactor theory, fluid mechanics, nonlinear dynamics [1,5,13,16,20,21,24]. In recent years, a variety of numerical methods are springing up based on the model, such as the collocation method with rationalized Haar functions [18], the homotopy perturbation method [2, 6, 23], direct method using triangular functions [4], a combined form of the Laplace transform method with the ADM [9,10], the HAM [22], the composite collocation method [17], a monotone method [3].

In this paper, our aim is to solve a general form of nonlinear Volterra-Fredholm integral equations using two approximate methods, namely: ADM and VIM. Moreover, we will study some new existence, uniqueness and convergence results of the solutions.

2. DESCRIPTION OF THE METHODS

In this section, we will describe some powerful methods have been focusing on the development of more advanced and efficient methods for solving integral equations such as the ADM [9,10,12,15,19], VIM [8,11,14,15].

2.1. Adomian Decomposition Method (ADM)

The ADM introduces the following expression:

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad (4)$$

for the solution $u(x)$ of (3), where the components $u_n(x)$ will be determined recurrently. The method defines the nonlinear function $G_1(u(t))$ and $G_2(u(t))$ by an infinite series of polynomials as

$$G_1(u(t)) = \sum_{n=0}^{\infty} A_n(t), \quad G_2(u(t)) = \sum_{n=0}^{\infty} B_n(t) \tag{5}$$

where A_n and B_n are so-called Adomian polynomials that represent the nonlinear term $G_1(u(t))$ and $G_2(u(t))$ and can be calculated for various classes of nonlinear operators according to specific algorithms set by [7, 15].

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda_1 \int_a^x K_1(x,t) \sum_{n=0}^{\infty} A_n(t)dt + \lambda_2 \int_a^b K_2(x,t) \sum_{n=0}^{\infty} B_n(t)dt. \tag{6}$$

As a result, the decomposition method introduces the recursive relation:

$$\begin{aligned} u_0(x) &= f(x) \\ u_n(x) &= \lambda_1 \int_a^x K_1(x,t)A_{n-1}dt + \lambda_2 \int_a^b K_2(x,t)B_{n-1}dt. \end{aligned} \tag{7}$$

Relation (7) will enable us to determine the components $u_n(x)$ recurrently for $n \geq 1$, and as a result, the series solution of $u(x)$ is readily obtained. Then, $u(x) = \sum_{i=0}^{n} u_i$ as the approximate solution.

2.2. Variational Iteration Method (VIM)

In this part, the extended the VIM is used to find approximate solutions of nonlinear Volterra-Fredholm equation (3), let $w(x)$ be a function such that $w'(x) = u(x)$, noting that $u(x)$ is continuous. Then we have,

$$w'(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)G_1(w'(t))dt + \lambda_2 \int_a^b K_2(x,t)G_2(w'(t))dt. \tag{8}$$

Consider

$$\lambda_1 \int_a^x K_1(x,t)F(w'(t))dt + \lambda_2 \int_a^b K_2(x,t)G(w'(t))dt, \tag{9}$$

as a restricted variation; we have the iteration sequence

$$\begin{aligned} w_{n+1} &= w_n + \int_a^x \lambda \Big[w_n'(s) - \lambda_1 \int_a^s K_1(s,t)G_1(w'(t))dt \\ &\quad -\lambda_2 \int_a^b K_2(s,t)G_2(w'(t))dt - f(s) \Big] ds. \end{aligned}$$

Taking the variation with respect to the independent variable w_n and noticing that $\delta w_n(0) = 0$, we get:

$$\delta w_{n+1} = \delta w_n + \lambda(s)\delta w_n \Big|_{s=x} - - \int_a^x \lambda'(s)\delta w_n ds = 0. \tag{10}$$

Then we apply the following stationary conditions:

$$1+\lambda(s)\Big|_{s=x}=0, \quad \lambda'(s)\Big|_{s=x}=0.$$

The general Lagrange multiplier, therefore, can be readily identified:

$$\lambda=-1,$$

and, as a result, we obtain the following iteration formula:

$$\begin{aligned} w_{n+1} &= w_n-\int_a^x \Big[w_n'(s)-\lambda_1\int_a^s K_1(s,t)G_1(w'(t))dt \\ &\quad -\lambda_2\int_a^b K_2(s,t)G_2(w'(t))dt-f(s)\Big]ds. \end{aligned}$$

Consequently, the approximate solution is given by

$$\lim_{n\longrightarrow\infty} u_n(x)=u(x).$$

3. UNIQUENESS AND CONVERGENCE RESULTS

In this section, we shall give uniqueness results of Eq. (1) and convergence of the methods and prove it. Before starting and proving the main results, we introduce the following hypotheses:

(A1) There exist two constants $M_1, M_2>0$ such that,

$$|\lambda_1 K_1(x,t)|\leq M_1, \;\; |\lambda_2 K_2(x,t)|\leq M_2, \quad \forall \;\; a\leq x,t\leq b.$$

(A2) Suppose the nonlinear operators $g_1(t,u(t)), g_2(t,u(t))$ are satisfied in Lipschitz conditions with

$$|g_1(t,u(t))-g_1(t,u^*(t))|\leq L_1|u-u^*|,$$
$$|g_2(t,u(t))-g_2(t,u^*(t))|\leq L_2|u-u^*|.$$

(A3) Consider $f(x)$ is bounded $\forall \;\; x\in[a,b]$.

Theorem 3.1 *Assume that (A1), (A2) and (A3) hold and if* $0<\alpha<1$, *where*

$$\alpha=(M_1L_1+M_2L_2)(b-a).$$

Then Eq.(1) *has a unique solution.*

Proof. Let u and u^* be two different solutions of Eq.(1) then

$$\begin{aligned}
|u-u^*| &= \Big|f(x)+\lambda_1\int_a^x K_1(x,t)g_1(t,u(t))dt+\lambda_2\int_a^b K_2(x,t)g_2(t,u(t))dt \\
&\quad -\Big(f(x)+\lambda_1\int_a^x K_1(x,t)g_1(t,u^*(t))dt \\
&\quad +\lambda_2\int_a^b K_2(x,t)g_2(t,u^*(t))dt\Big)\Big| \\
&= \Big|\lambda_1\int_a^x K_1(x,t)g_1(t,u(t))dt+\lambda_2\int_a^b K_2(x,t)g_2(t,u(t))dt \\
&\quad -\lambda_1\int_a^x K_1(x,t)g_1(t,u^*(t))dt-\lambda_2\int_a^b K_2(x,t)g_2(t,u^*(t))dt\Big| \\
&= \Big|\int_a^x \lambda_1K_1(x,t)g_1(t,u(t))dt-\int_a^x \lambda_1K_1(x,t)g_1(t,u^*(t))dt \\
&\quad +\int_a^b \lambda_2K_2(x,t)g_2(t,u(t))dt-\int_a^b \lambda_2K_2(x,t)g_2(t,u^*(t))dt\Big| \\
&= \Big|\int_a^x \lambda_1K_1(x,t)[g_1(t,u(t))-g_1(t,u^*(t))]dt \\
&\quad +\int_a^b \lambda_2K_2(x,t)[g_2(t,u(t))-g_2(t,u^*(t))]dt\Big| \\
&\leq \int_a^x |\lambda_1K_1(x,t)||g_1(t,u(t))-g_1(t,u^*(t))|dt \\
&\quad +\int_a^b |\lambda_2K_2(x,t)||g_2(t,u(t))-g_2(t,u^*(t))|dt \\
&\leq M_1L_1(b-a)|u-u^*|+M_2L_2(b-a)|u-u^*| \\
&= (M_1L_1+M_2L_2)(b-a)|u-u^*| \\
&= \alpha|u-u^*|,
\end{aligned}$$

from which we get $(1-\alpha)|u-u^*|\leq 0$. Since $0<\alpha<1$, then $|u-u^*|=0$ implies $u=u^*$ and this completes the proof.

Theorem 3.2 *If the series solution* $u(x)=\sum_{i=0}^{\infty}u_i(x)$ *obtained by the using ADM is convergent, then it converges to the exact solution of the Eq.*(1) *when* $0<\alpha<1$ *and* $\|u_1(x)\|<\infty$.

Proof. Denote as $(C[a,b],\|.\|)$ the Banach space of all continuous functions on $[a,b]$ with $|u_1(x)|\leq\infty$ for all x in $[a,b]$.

Firstly, let s_n and s_m be arbitrary partial sums with $n \geq m$. We are going to prove that $s_n = \sum_{i=0}^{n} u_i(x)$ is a Cauchy sequence in this Banach space:

$$\begin{aligned} \|s_n - s_m\| &= \max_{\forall x \in J} |s_n - s_m| \\ &= \max_{\forall x \in J} \Big|\sum_{i=0}^{n} u_i(x) - \sum_{i=0}^{m} u_i(x)\Big| \\ &= \max_{\forall x \in J} \Big|\sum_{i=m+1}^{n} u_i(x)\Big| \\ &= \max_{\forall x \in J} \Big| \sum_{i=m+1}^{n} \Big[\lambda_1 \int_a^x K_1(x,t)A_i(t)dt + \lambda_2 \int_a^b K_2(x,t)B_i(t)dt\Big] \Big| \\ &= \max_{\forall x \in J} \Big| \int_a^x \lambda_1 K_1(x,t) \sum_{i=m}^{n-1} A_i(t)dt + \int_a^b \lambda_2 K_2(x,t) \sum_{i=m}^{n-1} B_i(t)dt) \Big|. \end{aligned}$$

From (5), we have

$$\sum_{i=m}^{n-1} A_i = g_1(t, s_{n-1}) - g_1(t, s_{m-1}), \ \sum_{i=m}^{n-1} B_i = g_2(t, s_{n-1}) - g_2(t, s_{m-1}).$$

So,

$$\begin{aligned} \|s_n - s_m\| &= \max_{\forall x \in J} \Big| \int_a^x \lambda_1 K_1(x,t)(g_1(t, s_{n-1}) - g_1(t, s_{m-1}))dt \\ &\quad + \int_a^b \lambda_2 K_2(x,t)(g_2(t, \underline{s}_{n-1}) - g_2(t, s_{m-1}))dt \Big| \\ &\leq \max_{\forall x \in J} \Big(\int_a^x |\lambda_1 K_1(x,t)||(g_1(t, s_{n-1}) - g_1(t, s_{m-1}))|dt \\ &\quad + \int_a^b |\lambda_2 K_2(x,t)||(g_2(t, s_{n-1}) - g_2(t, s_{m-1}))|dt \Big) \\ &\leq M_1 L_1 \|s_{n-1} - s_{m-1}\|(b-a) + M_2 L_2 \|s_{n-1} - s_{m-1}\|(b-a) \\ &= (L_1 M_1 + M_2 L_2)(b-a)\|s_{n-1} - s_{m-1}\| \\ &= \alpha \|s_{n-1} - s_{m-1}\|. \end{aligned}$$

Let $n = m + 1$, then,

$$\|s_n - s_m\| \leq \alpha \|s_m - s_{m-1}\| \leq \alpha^2 \|s_{m-1} - s_{m-2}\| \leq \cdots \leq \alpha^m \|s_1 - s_0\|.$$

So,

$$\begin{aligned} \|s_n - s_m\| &\leq \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \cdots + \|s_n - s_{n-1}\| \\ &\leq [\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1}]\|s_1 - s_0\| \\ &\leq \alpha^m [1 + \alpha + \alpha^2 + \cdots + \alpha^{n-m-1}]\|s_1 - s_0\| \\ &\leq \alpha^m \Big(\frac{1 - \alpha^{n-m}}{1 - \alpha}\Big)\|u_1(x)\|. \end{aligned}$$

Since $0 < \alpha < 1$, we have $(1 - \alpha^{n-m}) < 1$, then

$$\|s_n - s_m\| \leq \frac{\alpha^m}{1-\alpha}\|u_1(x)\|.$$

But $|u_1(x)| < \infty$, so, as $m \longrightarrow \infty$, then $\|s_n - s_m\| \longrightarrow 0$. We conclude that s_n is a Cauchy sequence in $C[a, b]$. Then the series is convergence and the proof is complete.

Theorem 3.3 *If the solution* $u(x) = \lim_{n\to\infty} u_n(x)$ *obtained by using VIM is convergent, then it converges to the exact solution of the Eq.*(1) *with* $0 < \alpha < 1$.

Proof. The iteration formula as follows:

$$\begin{aligned} u_{n+1}(x) &= u_n(x) - \int_0^x [u_n(s) - f(s) - \lambda_1 \int_a^s K_1(s,t) g_1(t, u_n(t)) dt \\ &\quad -\lambda_2 \int_a^b K_2(s,t) g_2(t, u_n(t)) dt] ds, \end{aligned} \tag{11}$$

we can write

$$\begin{aligned} u(x) &= u(x) - \int_0^x [u(s) - f(s) - \lambda_1 \int_a^s K_1(s,t) g_1(t, u(t)) dt \\ &\quad -\lambda_2 \int_a^b K_2(s,t) g_2(t, u(t)) dt] ds. \end{aligned} \tag{12}$$

By subtracting Eq.(11) from Eq.(12),

$$\begin{aligned} u_{n+1}(x) - u(x) &= u_n(x) - u(x) - \int_0^x [u_n(s) - u(s) \\ &\quad -\lambda_1 \int_a^s K_1(s,t)[g_1(t, u_n(t)) - g_1(t, u(t))] dt \\ &\quad -\lambda_2 \int_a^b K_2(s,t)[g_2(t, u_n(t)) - g_2(t, u(t))] dt] ds. \end{aligned}$$

If we set, $e_{n+1}(x) = u_{n+1}(x) - u(x)$, and $e_n(x) = u_n(x) - u(x)$ then

$$\begin{aligned} e_{n+1}(x) &= e_n(x) - \int_0^x [e_n(s) - \lambda_1 \int_a^s K_1(s,t)[g_1(t, u_n(t)) - g_1(t, u(t))] dt \\ &\quad -\lambda_2 \int_a^b K_2(s,t)[g_2(t, u_n(t)) - g_2(t, u(t))] dt] ds + e_n(x) - e_n(x, 0) \\ &\leq e_n(x)(1 - (b-a)(M_1 L_1 + M_2 L_2)) \\ &= (1-\alpha) e_n(x), \end{aligned}$$

therefore,

$$\begin{aligned} \|e_{n+1}\| &= \max_{\forall x \in J} |e_{n+1}| \\ &\leq (1-\alpha) \max_{\forall x \in J} |e_n| \\ &= \|e_n\|, \end{aligned}$$

since $0 < \alpha < 1$, then $\|e_n\| \longrightarrow 0$. So, the series converges and the proof is complete.

4. NUMERICAL EXAMPLE

In this section, we present the semi-analytical techniques based on ADM and VIM to solve nonlinear Volterra-Fredholm integral equations.

Example 1. Let us consider the nonlinear Volterra-Fredholm integral equation:

$$\frac{x^4}{8e^2} + 0.0450658 + 0.300274x - \int_{0.1}^{x} \frac{[u(t)]^2}{2} dt - \int_{0.1}^{0.5} (x+t)(1+[u(t)]^2) dt = 0. \quad (13)$$

The exact solution is $u(x) = xe^{-x}$.

Table 1: Numerical Results of the Example 1.

x	Exact	$\text{ADM}_{n=12}$	$\text{VIM}_{n=8}$
0.1	0.11051709	0.10477329	0.10987929
0.2	0.24428055	0.23800425	0.24361525
0.3	0.40495764	0.39827314	0.40423694
0.4	0.59672988	0.58987258	0.59596048
0.5	0.82436064	0.81704804	0.82363674

Table 2: Errors Results of the Example 1.

x	Errors (ADM,n=12)	Errors (VIM,n=8)
0.1	0.0057438	0.0006378
0.2	0.0062763	0.0006653
0.3	0.0066845	0.0007207
0.4	0.0068573	0.0007694
0.5	0.0073126	0.0007239

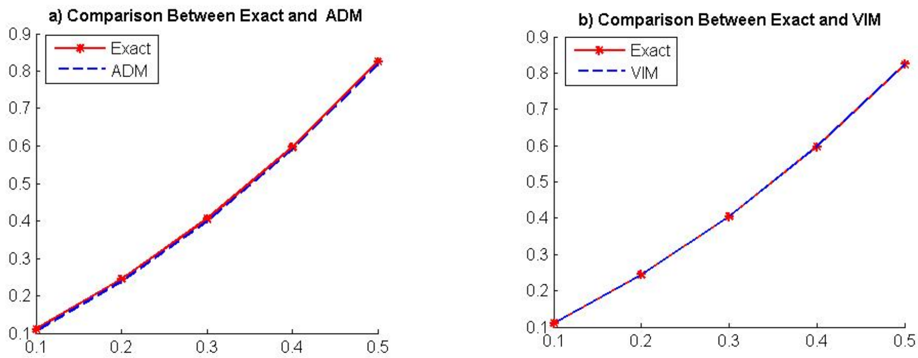

Figure 1: Numerical Results of the Example 1.

5. CONCLUSION

In this work, the ADM and VIM are used to solve the nonlinear Volterra-Fredholm integral equations. We described the methods, used them on one test problem, and compared the results with their exact solutions in order to demonstrate the validity and applicability of the methods. Moreover, only a small number of iterations are needed to obtain a satisfactory result. The given numerical example support this claim. The above tables show a comparison between the exact solution and the approximated solutions of the illustrative example, by using ADM and VIM with different iterations and number of terms. One advantage of VIM is that the initial solution can be freely chosen with some unknown parameters. An interesting point about this method is that with few number of iterations, or even in some cases with only one iteration, it can produce a very accurate approximate solution. The error convergence to zero if the iterations of used terms increases. For this purpose, we showed that the VIM is more rapid convergence than ADM.

REFERENCES

[1] I. Alasadi and A. Hamoud. Existence and stability results for fractional Volterra-Fredholm integro-dierential equation with mixed conditions. *Advances in Dynamical Systems and Applications*, **16**(1) (2021), 217–236.

[2] P. Bhadane, K. Ghadle and A. Hamoud. Approximate solution of fractional Black-Schole's European option pricing equation by using ETHPM. *Nonlinear Functional Analysis and Applications*, **25**(2) (2020), 331–344.

[3] M. Al-Mdallal. Monotone iterative sequences for nonlinear integro-differential equations of second order. *Nonlinear Anal. Real World Appl.* **12**(6) (2011), 3665–3673.

[4] E. Babolian, Z. Masouri, and S. Hatamzadeh-Varmazyar. Numerical solution of nonlinear Volterra-Fredholm integro-differential equations via direct method using triangular functions. *Comput. Math. Appl.* **58**(2) (2009), 239–247.

[5] M. Bani Issa, A. Hamoud and K. Ghadle. Numerical solutions of fuzzy integro-differential equations of the second kind. *Journal of Mathematics and Computer Science*, **23** (2021),67–74.

[6] Sh. Behzadi. Homotopy approximation technique for solving nonlinear Vollterrs-Fredholm integral equations of the first kind. *Int. J. Ind. Math.* **6**(4) (2014), 315–320.

[7] N. Bildik and M. Inc. Modified decomposition method for nonlinear Volterra-Fredholm integral equations. *Chaos Solitons Fractals*, **33** (2007),308–313.

[8] L. Dawood, A. Sharif and A. Hamoud. Solving higher-order integro differential equations by VIM and MHPM. *International Journal of Applied Mathematics*, **33**(2) (2020), 253–264.

[9] A. Hamoud and K. Ghadle. Modified Adomian decomposition method for solving fuzzy Volterra-Fredholm integral equations. *J. Indian Math. Soc.*, **85**(1-2) (2018), 52–69.

[10] A. Hamoud, A. Azeez and K. Ghadle. A study of some iterative methods for solving fuzzy Volterra-Fredholm integral equations. *Indonesian Journal of Electrical Engineering and Computer Science*, **11**(3) (2018), 1228–1235.

[11] A. Hamoud, L. Dawood, K. Ghadle and S. Atshan. Usage of the modified variational iteration technique for solving Fredholm integro-differential equations. *International Journal of Mechanical and Production Engineering Research and Development*, **9**(2) (2019), 895–902.

[12] A. Hamoud, N. Mohammed and K. Ghadle. A study of some effective techniques for solving Volterra-Fredholm integral equations. *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, **26** (2019), 389–406.

[13] A. Hamoud. Uniqueness and stability results for Caputo fractional Volterra-Fredholm integro-differential equations. *J. Sib. Fed. Univ. Math. Phys.*, **14**(3) (2021), 313–325.

[14] A. Hamoud. Existence and uniqueness of solutions for fractional neutral Volterra-Fredholm integro-differential equations. *Advances in the Theory of Nonlinear Analysis and its Application*, **4**(4) (2020), 321–331.

[15] A. Jerri. *Introduction to Integral Equations with Applications*, New York, Wiley, 1999.

[16] S. Jiang and V. Rokhlin. Second kind integral equations for the classical potential theory on open surfaces II. *J. Comput. Phys.* **195**(1) (2004), 1–16.

[17] R. Marzban, R. Tabrizidooz and M. Razzaghi. A composite collocation method for the nonlinear mixed Volterra-Fredholm-Hammerstein integral equations. *Commun. Nonlinear Sci. Numer. Simul.* **16**(3) (2011), 1186–1194.

[18] Y. Ordokhani and M. Razzaghi. Solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via a collocation method and rationalized Haar functions. *Appl. Math. Lett.* **21**(1) (2008), 4–9.

[19] B. Pachpatte. On mixed Volterra-Fredholm type integral equations, *Indian J. Pure Appl. Math.* **17**(4) (1986), 488–496.

[20] P. Schiavone, C. Constanda and A. Mioduchowski. *Integral Methods in Science and Engineering*, Boston: Birkhauser, 2012.

[21] I. Smetanin. On an integral equation for axially-symmetric problems in the case of an elastic body containing an inclusion. *J. Appl. Math. Mech.* **55**(3) (1991), 371–375.

[22] A. Shidfar, A. Molabahrami and A. Babaei. A series solution of the nonlinear Volterra and Fredholm integro-differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **15**(2) (2010), 205–215.

[23] A. Yildirim. Solution of BVPs for fourth-order integro-differential equations by using homotopy perturbation method. *Comput. Math. Appl.* **56**(12) (2008), 3175–3180.

[24] M. F. Younis, A. M. Abed and A. A. Hamoud. Existence and uniqueness results for bvp of nonlinear fractional Volterra-Fredholm integro-differential equation. *Advances in Dynamical Systems and Applications*, **16**(2) (2021) 457–467.