Polynomial Sharing and Uniqueness of Differential-Difference Polynomials of L-functions

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Abstract

In this paper, we study value distributions and uniqueness problems of differential-difference polynomials of L-functions. Considering polynomial sharing of certain differential-difference polynomials of an L-function with that of a meromorphic function we prove a uniqueness theorem which improve and generalize some earlier results due to Hao, Chen [4], Zhu, Chen [16], Mandal, Datta [10] and Datta, Mandal [2].

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1. INTRODUCTION

For the last 150 years the most important open problem in pure mathematics is considered to be the Riemann hypothesis and its extension to the general classes of L-functions. L-functions are most important objects in the modern number theory. Let a function \( L \) be defined by the Dirichlet series \( L(z) = \sum_{n=1}^{\infty} a(n)/n^z \) with \( a_1 = 1 \) satisfying the axioms (i) \( a(n) \ll n^\epsilon \), for every \( \epsilon > 0 \), (ii) there exists an integer \( k \geq 0 \)
such that \((z - 1)^k L(z)\) is a finite order entire function, (iii) every L-function satisfies the functional equation

\[
\lambda_L(z) = \omega \lambda_L(1 - \bar{z}),
\]

where

\[
\lambda_L(z) = L(z)Q \prod_{i=1}^{k} \Gamma(\lambda_i z + \nu_i)
\]

with positive real numbers \(Q, \lambda_i\) and complex numbers \(\nu_i, \omega\) with \(Re \nu_i \geq 0\) and \(|\omega| = 1\) and (iv) \(L(z)\) satisfies \(L(z) = \prod_p L_p(z)\), where \(L_p(z) = \exp(\sum_{k=1}^{\infty} b(p^k) / p^{kz})\) with coefficients \(b(p^k)\) satisfying \(b(p^k) \ll p^{\theta k}\) for some \(\theta < 1/2\) and \(p\) denotes prime number. Then \(L\) is said to be an L-function in the Selberg class. If \(L\) satisfy axioms (i)-(iii) then \(L\) is said to be an L-function in the extended Selberg class. Henceforth by an L-function we always mean an L-function in the extended Selberg class.

In this paper, we concentrate our attention on the uniqueness problems of differential-difference polynomial of L-functions. We use the standard definitions and notations of value distribution theory [5].

2. PRELIMINARIES

Let \(\alpha \in \mathbb{C} \cup \{\infty\}\) and \(\xi, \psi\) be meromorphic functions in the complex plane. The hyper order \(\rho_2(\xi)\) of \(\xi\) is defined by \(\rho_2(\xi) = \lim \sup_{r \to \infty} \frac{\log \log T(r, \xi)}{\log r}\). We denote by \(S(r, \xi)\) any function satisfying \(S(r, \xi) = o(T(r, \xi))\) as \(r \to \infty\), outside a possible exceptional set of finite linear measure.

**Definition 2.1.** [6, 7]. Let \(\xi\) and \(\psi\) be meromorphic functions defined in the complex plane and \(n\) be an integer \((\geq 0)\) or infinity. For \(\alpha \in \mathbb{C} \cup \{\infty\}\) we denote by \(E_n(\alpha; \xi)\) the set of all zeros of \(\xi - \alpha\) where a zero of multiplicity \(k\) is counted \(k\) times if \(k \leq n\) and \(n + 1\) times if \(k > n\). If \(E_n(\alpha; \xi) = E_n(\alpha; \psi)\), we say that \(\xi, \psi\) share the value \(\alpha\) with weight \(n\). We say \(\xi, \psi\) share \((\alpha, n)\) to mean that \(\xi, \psi\) share the value \(\alpha\) with weight \(n\).

**Definition 2.2.** [10]. Let \(\xi\) be a meromorphic function defined in the complex plane and \(P(z)\) be a polynomial or a small function of \(\xi\). Then we denote by \(E_m(P; \xi), \overline{E}_m(P; \xi)\) and \(E_m; \xi)\) the sets \(E_{m}((0; \xi - P), \overline{E}_{m}(0; \xi - P)\) and \(E_{m}(0; \xi - P)\) respectively. We write \(\xi, \psi\) share \((P, n)\) to mean that \(\xi - P, \psi - P\) share the value \(0\) with weight \(n\).
Liu, Li and Yi [9] in 2017 proved the following uniqueness theorem.

**Theorem 2.1.** [9] Let $L$ be an $L$-function and $\xi$ be a nonconstant meromorphic function. If $j \geq 1$, $k \geq 1$ be integers such that $j > 3k + 6$ and $\{\xi^j\}^{(k)}(z)$, $\{L^j\}^{(k)}(z)$ share $(z, \infty)$ then $\xi \equiv \alpha L$ for some nonconstant $\alpha$ satisfying $\alpha^j = 1$.


**Theorem 2.2.** [4] Let $\xi$ be a nonconstant meromorphic function and $L$ be an $L$-function such that $[\xi^n(\xi - 1)^m]^{(\tau)}$ and $[L^n(L - 1)^m]^{(\tau)}$ share $(1, \infty)$, where $n, m, \tau \in \mathbb{Z}^+$. If $n > m + 3\tau + 6$ and $\tau \geq 2$, then $\xi \equiv L$ or $\xi^n(\xi - 1)^m \equiv L^n(L - 1)^m$.

**Theorem 2.3.** [4] Let $\xi$ be a nonconstant meromorphic function and $L$ be an $L$-function such that $[\xi^n(\xi - 1)^m]^{(\tau)}$ and $[L^n(L - 1)^m]^{(\tau)}$ share $(1, 0)$, where $n, m, \tau \in \mathbb{Z}^+$. If $n > 4m + 7\tau + 11$ and $\tau \geq 2$, then $\xi \equiv L$ or $\xi^n(\xi - 1)^m \equiv L^n(L - 1)^m$.

Using truncated sharing in 2019 W. Q. Zhu and J. F. Chen [16] proved the following theorem.

**Theorem 2.4.** [16] Let $L$ be an $L$-function and $\xi$ be a transcendental meromorphic function defined in the complex plane $\mathbb{C}$. Also let $n, k(\geq 2)$, $l(\geq 2)$ be positive integers such that $n \geq 7k + 17$. If $E(l)(1, (\xi^n(\xi - 1))^{(k)}) = E(l)(1, (L^n(L - 1))^{(k)})$ then $\xi \equiv L$.

**Definition 2.3.** [8]. Let $\xi$ be a meromorphic function defined in the complex plane. Let $k \geq 1$ be an integer and $\alpha \in \mathbb{C} \cup \{\infty\}$. By $N(r, \alpha; \xi | \leq k)$ we denote the counting function of the $\alpha$ points of $\xi$ with multiplicity not greater than $k$ and by $\overline{N}(r, \alpha; \xi | \leq k)$ the reduced counting function. Also by $N(r, \alpha; \xi | \geq k)$ we denote the counting function of the $\alpha$ points of $\xi$ with multiplicity not less than $k$ and by $\overline{N}(r, \alpha; \xi | \geq k)$ the reduced counting function. We define

$$N_k(r, \alpha; \xi) = \overline{N}(r, \alpha; \xi) + \overline{N}(r, \alpha; \xi | \geq 2) + \cdots + \overline{N}(r, \alpha; \xi | \geq k).$$

Considering small function sharing in 2020 Mandal and Datta [10] proved the following theorem.
Theorem 2.5. [10]. Let \( L \) be a nonconstant \( L \)-function and \( \rho \) be a small function of \( L \) such that \( \rho \neq 0, \infty \). If \( E_0(\rho; L) = E_0(\rho; (L^m)^{(k)}) \), \( E_2(\rho; L) = E_2(\rho; (L^m)^{(k)}) \) and
\[
2N_{2+k}(r, 0; L^m) \leq (\sigma + o(1))T(r, L),
\]
where \( m \geq 1, k \geq 1 \) are integers and \( 0 < \sigma < 1 \), then \( L \equiv (L^m)^{(k)} \).

Using weighted sharing Datta and Mandal [2] proved the following theorem.

Theorem 2.6. [2]. Let \( f \) be a nonconstant meromorphic function and \( L \) be a nonconstant \( L \)-function. If \( E_0(0, f) = E_0(0, L) \), \( E_1(1, f) = E_1(1, L) \) and \( N(r, 0; f) + N(r, \infty; f) = S(r, f) \) then either \( L \equiv f \) or \( T(r, L) = N(r, 0; L \mid \leq 2) + S(r, L) \) and
\[
T(r, f) = N(r, 0; L' \mid \leq 1) + S(r, L).
\]

Now the following questions come naturally.

Question 2.1. If we consider polynomial sharing in theorem 2.1, 2.2, 2.3, 2.4, 2.5 and 2.6 then what will be the results?

Question 2.2. If we consider differential-difference polynomials in place of differential polynomials in theorem 2.1, 2.2, 2.3, 2.4 and 2.5 then what will be the results?

Definition 2.4. [6]. Let two nonconstant meromorphic functions \( \xi \) and \( \psi \) share a value \( \alpha \) \( IM \). We denote by \( \overline{N}_s(r, \alpha; \xi, \psi) \) the counting function of the \( \alpha \)-points of \( \xi \) and \( \psi \) with different multiplicities, where each \( \alpha \)-point is counted only once.

Clearly \( \overline{N}_s(r, \alpha; \xi, \psi) = \overline{N}_s(r, \alpha; \psi, \xi) \).

Definition 2.5. Let two nonconstant meromorphic functions \( \xi \) and \( \psi \) share a value \( \alpha \) \( IM \). We denote by \( \overline{N}(r, \alpha; | \xi > \psi) \) the counting function of the \( \alpha \)-points of \( \xi \) and \( \psi \) with multiplicities with respect to \( \xi \) is greater than the multiplicities with respect to \( \psi \), where each \( \alpha \)-point is counted once only.

Definition 2.6. Let two nonconstant meromorphic functions \( \xi \) and \( \psi \) share a value \( \alpha \) \( IM \). We denote by \( \overline{N}_E(r, \alpha; \xi, | \psi > m) \) the counting function of the \( \alpha \)-points of \( \xi \) and \( \psi \) with multiplicities greater than \( m \) and the multiplicities with respect to \( \xi \) is equal to the multiplicities with respect to \( \psi \), where each \( \alpha \)-point is counted once only.

Definition 2.7. [8]. Let \( \xi \) be a meromorphic function defined in the complex plane and \( P \) be a small function of \( \xi \) or a polynomial of \( z \). Then we denote by \( N(r, P; \xi \mid \leq k) \), \( \overline{N}(r, P; \xi \mid \leq k) \), \( N(r, P; \xi \mid > k) \), \( \overline{N}(r, P; \xi \mid > k) \), \( N_k(r, P; \xi) \) etc. the counting functions \( N(r, 0; \xi - P \mid \leq k) \), \( \overline{N}(r, 0; \xi - P \mid \leq k) \), \( N(r, 0; \xi - P \mid > k) \), \( \overline{N}(r, 0; \xi - P \mid > k) \), \( N_k(r, 0; \xi - P) \) etc. respectively.
3. MAIN RESULTS

Let $L$ be a nonconstant $L$-function, $\xi$ be a transcendental meromorphic function and $P(z)$ be a polynomial of $z$. Also let $\tau, n, \eta, \mu_j (j = 1, 2, \ldots, \eta)$ be positive integers and $\omega_j \in \mathbb{C} - \{0\}$ ($j = 1, 2, \ldots, \eta$) be distinct constants. Henceforth we denote by $\phi, \psi, \Phi, \Psi$ the following functions $\phi(z) = \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}, \psi(z) = \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}, \Phi(z) = \xi(z)^n [\phi(z)]^{(\tau)}$ and $\Psi(z) = \frac{\xi(z)^n [\phi(z)]^{(\tau)}}{P(z)}$.

Using the concept of weighted sharing we try to solve Questions 2.1, 2.2 and prove the following theorem.

**Theorem 3.1.** Let $L$ be a nonconstant $L$-function and $\xi$ be a transcendental meromorphic function such that $\rho_2(L) < 1$, $\rho_2(\xi) < 1$ and $\xi, L$ share $(\infty, 0)$. If $L(z)^n[\psi(z)]^{(\tau)}$ and $\xi(z)^n[\phi(z)]^{(\tau)}$ share $(P(z), l)$, where $0 \leq l < \infty$ and $P(z)$ is a polynomial of $z$, then one of the following holds

(i) $L(z)^n[\psi(z)]^{(\tau)} \equiv \xi(z)^n[\phi(z)]^{(\tau)}$

(ii) $L = e^{a(z)}$ and $\xi = e^{b(z)}$, where $a(z)$ and $b(z)$ are entire functions

if

(i) $l = 0$ and $n > \max\{\lambda + \eta(5\tau + 7) + 7, c_1, c_2\}$

(ii) $l = 1$ and $n > \max\{\lambda + \frac{1}{2}(\eta(5\tau + 9) + 7), c_1, c_2\}$

(iii) $l \geq 2$ and $n > \max\{\lambda + \eta(2\tau + 4) + 4, c_1, c_2\}$, where $c_1 = \sum_{j=1}^{\eta} a_j\mu_j$ and $c_2 = \sum_{j=1}^{\eta} b_j\mu_j$, $a_j$ and $b_j$ denotes maximum orders of zeros of $\xi(z + \omega_j)$ and $L(z + \omega_j)$ respectively for $j = 1, 2, \ldots, \eta$.

4. LEMMAS

In this section we present some lemmas which will be needed in the proof of our results. Henceforth we denote by $\Omega_{\Phi, \Psi}$ the function defined by

$$\Omega_{\Phi, \Psi} = \frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi - 1} - \frac{\Psi''}{\Psi'} + \frac{2\Psi'}{\Psi - 1}$$
Lemma 4.1. [12]. Let $L$ be an L-function with degree $q$. Then

$$T(r, L) = \frac{q}{\pi} r \log r + O(r).$$

Lemma 4.2. [10]. Let $L$ be an L-function. Then $N(r, \infty; L) = S(r, L) = O(\log r)$.

Lemma 4.3. Let $\xi$ be a transcendental meromorphic function and $L$ be an L-function. If $\xi$ and $L$ share $(\infty, 0)$ then $N(r, \infty; \xi) = O(\log r) = S(r, \xi)$ and $N(r, \infty; \xi) = O(\log r) = S(r, L)$.

Proof. Since $\xi$ and $L$ share $(\infty, 0)$ therefore $\xi$ has finitely many poles. Hence $N(r, \infty; \xi) = O(\log r)$. Again since $\xi$ and $L$ are transcendental meromorphic functions therefore $N(r, \infty; \xi) = O(\log r) = S(r, \xi)$ and $N(r, \infty; \xi) = O(\log r) = S(r, L)$. This completes the proof. \qed

Lemma 4.4. [15]. Let $\xi(z) = \frac{\alpha_0 + \alpha_1 z + \ldots + \alpha_n z^n}{\beta_0 + \beta_1 z + \ldots + \beta_m z^m}$ be a nonconstant rational function defined in the complex plane $\mathbb{C}$, where $\alpha_0, \alpha_1, \ldots, \alpha_n (\neq 0)$ and $\beta_0, \beta_1, \ldots, \beta_m (\neq 0)$ are complex constants. Then

$$T(r, \xi) = \max\{m, n\} \log r + O(1).$$

Lemma 4.5. [13]. Let $\xi$ be a transcendental meromorphic function of hyper order $\rho_2(\xi) < 1$. Then for any $\alpha \in \mathbb{C} - \{0\}$

$$T(r, \xi(z + \alpha)) = T(r, \xi(z)) + S(r, \xi(z))$$

$$N(r, \infty; \xi(z + \alpha)) = N(r, \infty; \xi(z)) + S(r, \xi(z))$$

$$N(r, 0; \xi(z + \alpha)) = N(r, 0; \xi(z)) + S(r, \xi(z))$$

Lemma 4.6. [1] Let $f$ and $g$ be two nonconstant meromorphic functions sharing $(1, l)$ and $(\infty, 0)$ where $2 \leq l < \infty$. If $\Omega_{f,g} \neq 0$ then

$$T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; f, g) - m(r, 1; g) - N_E(r, 1; f, g \mid > 3) - \overline{N}(r, 1; g \mid > f) + S(r, f) + S(r, g)$$

$$T(r, g) \leq N_2(r, 0; f) + N_2(r, 0; g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; f, g) - m(r, 1; f) - N_E(r, 1; g, f \mid > 3) - \overline{N}(r, 1; f \mid > g) + S(r, f) + S(r, g)$$
Lemma 4.7. [11] Let $f$ and $g$ be two nonconstant meromorphic functions sharing $(1, 1)$ and $(\infty, 0)$. If $\Omega_{f,g} \neq 0$ then

\[
T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + \frac{3}{2}N(r, \infty; f) + \overline{N}(r, \infty; f) + \overline{N}_*(r, \infty; f, g) + \frac{1}{2}N(r, 0; f) + S(r, f) + S(r, g)
\]

\[
T(r, g) \leq N_2(r, 0; f) + N_2(r, 0; g) + \frac{3}{2}N(r, \infty; f) + \overline{N}(r, \infty; f) + \overline{N}_*(r, \infty; f, g) + \frac{1}{2}N(r, 0; g) + S(r, f) + S(r, g)
\]

Lemma 4.8. [11] Let $f$ and $g$ be two nonconstant meromorphic functions sharing $(1, 0)$ and $(\infty, 0)$. If $\Omega_{f,g} \neq 0$ then

\[
T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + 3N(r, \infty; f) + 2\overline{N}(r, \infty; f) + \overline{N}_*(r, \infty; f, g) + 2N(r, 0; f) + \overline{N}(r, 0; g) + S(r, f) + S(r, g)
\]

\[
T(r, g) \leq N_2(r, 0; f) + N_2(r, 0; g) + 3N(r, \infty; f) + 2\overline{N}(r, \infty; f) + \overline{N}_*(r, \infty; f, g) + 2\overline{N}(r, 0; f) + S(r, f) + S(r, g)
\]

Lemma 4.9. [14] Let $f$ be a nonconstant meromorphic function and $k$, $p$ be two positive integers. Then

\[
T(r, f^{(k)}) \leq T(r, f) + k\overline{N}(r, \infty; f) + S(r, f)
\]

\[
N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f)
\]

\[
N_p(r, 0; f^{(k)}) \leq N_{p+k}(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f)
\]

\[
N(r, 0; f^{(k)}) \leq N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f)
\]

Lemma 4.10. [3] Let $\xi$ be a transcendental meromorphic function of hyper order $\rho_2(\xi) < 1$. Then

\[
(n - \lambda)T(r, \xi) + S(r, \xi) \leq T(r, \xi^n\phi) \leq (n + \lambda)T(r, \xi) + S(r, \xi)
\]

Lemma 4.11. Let $\xi$ be a transcendental meromorphic function of hyper order $\rho_2(\xi) < 1$. If $\xi$ and an $L$-function $L$ share $(\infty, 0)$ then

\[
(n - \lambda)T(r, \xi) + S(r, \xi) \leq T(r, \xi^n\phi^{(r)}) \leq (n + \lambda)T(r, \xi) + S(r, \xi)
\]
Proof. Since $\xi$ and $L$ share $(\infty, 0)$ therefore by lemma 4.3 we get $N(r, \infty; \xi) = S(r, \xi)$.
Hence by lemma 4.5 we have

$$T(r, \xi^n \phi^{(r)}) \leq T(r, \xi^n) + T(r, \phi^{(r)}) + S(r, \xi)$$
$$\leq nT(r, \xi) + T(r, \phi) + \tau N(r, \infty, \phi) + S(r, \xi)$$
$$\leq (n + \lambda)T(r, \xi) + S(r, \xi). \quad (4.1)$$

Also by lemma 4.5 we have

$$nT(r; \xi) = T(r, \xi^n) + S(r, \xi)$$
$$\leq T(r, \xi^n \phi^{(r)}) + S(r, \xi)$$
$$\leq T(r, \xi^n \phi^{(r)}) + T(r, \phi^{(r)}) + S(r, \xi)$$
$$\leq T(r, \xi^n \phi^{(r)}) + T(r, \phi) + \tau N(r, \infty, \phi) + S(r, \xi)$$
$$\leq T(r, \xi^n \phi^{(r)}) + \lambda T(r, \xi) + S(r, \xi). \quad (4.2)$$

We get from (4.2)

$$(n - \lambda)T(r, \xi) \leq T(r, \xi^n \phi^{(r)}) + S(r, \xi). \quad (4.3)$$

Hence we get from (4.1) and (4.3)

$$(n - \lambda)T(r, \xi) + S(r, \xi) \leq T(r, \xi^n \phi^{(r)}) \leq (n + \lambda)T(r, \xi) + S(r, \xi)$$

This completes the proof of the lemma.

Lemma 4.12. Let $L$ be a nonconstant $L$-function and $\xi$ be a transcendental meromorphic function such that $\rho_2(L) < 1$, $\rho_2(\xi) < 1$ and $\xi, L$ share $(\infty, 0)$. Also let $L(z)^n [\psi(z)]^{(r)}$ and $\xi(z)^n [\phi(z)]^{(r)}$ share $(P(z), 0)$ and, where $P(z)$ is a polynomial of $z$. If $\Omega_{\Phi, \Psi} \neq 0$ then $n \leq \lambda + \eta (5\tau + 7) + 7$.

Proof. Since $\xi, L$ share $(\infty, 0)$ and $\xi(z)^n \phi(z)^{(r)}, L(z)^n \psi(z)^{(r)}$ share $(P(z), 0)$ therefore $\Phi$ and $\Psi$ share $(1, l)$ except zeros of $P(z)$ and share $(\infty, 0)$.
By lemma 4.1 it is clear that $L$ is a transcendental meromorphic function.
Since $L$ and $\xi$ are transcendental meromorphic functions therefore $P$ is a small function of $L$ and $\xi$. 

\[\square\]
Hence by lemma 4.2, lemma 4.3 and lemma 4.8 we have

\[ T(r, L^n \psi^{(r)}) = T(r, \Psi) + S(r, L) \]
\[ \leq N_2(r, 0; \Phi) + N_2(r, 0; \Psi) + 3N(r, \infty; \Psi) + 2N(r, \infty; \Phi) + N_s(r, \infty; \Phi) + 2N(r, 0; \Psi) + N(r, 0; \Phi) + S(r, \Phi) + S(r, L) \]
\[ \leq N_2(r, 0; \xi^n) + N_2(r, 0; \psi^{(r)}) + N_2(r, 0; L^n) + N_2(r, 0; \psi^{(r)}) + 2N(r, 0; L^n) + 2N(r, 0; \psi^{(r)}) + 2N(r, 0; \xi^n) + N(r, 0; \phi^{(r)}) + S(r, \Phi_1) + S(r, L) \]
\[ \leq 2T(r, L) + N_2(r, 0; \psi^{(r)}) + 2N(r, 0; L^n) + 2N(r, 0; \psi^{(r)}) + 2T(r, \xi) + N_2(r, 0; \phi^{(r)}) + N(r, 0; \xi^n) + N(r, 0; \phi^{(r)}) + S(r, \Phi_1) + S(r, L) \]
\[ \leq 2T(r, L) + T(r, L^n \psi^{(r)}) - T(r, \psi) + N_2(r, 0; \psi^{(r)}) + N_2(r, 0; \phi^{(r)}) + N(r, 0; \xi^n) + N(r, 0; \phi^{(r)}) + S(r, \Phi_1) + S(r, L) \]
\[ \leq 2T(r, L) + T(r, L^n \psi^{(r)}) - T(r, L^n \psi) + \eta(2 + \tau)T(r, L) + 2N(r, 0; L^n) + 2N(r, 0; \psi^{(r)}) + 2T(r, \xi) + \eta(2 + \tau)T(r, \xi) + N(r, 0; \xi^n) + N(r, 0; \phi^{(r)}) + S(r, \xi) + S(r, L) \]
\[ \leq 2T(r, L) + T(r, L^n \psi^{(r)}) - T(r, L^n \psi) + \eta(2 + \tau)T(r, L) + 2T(r, L) + 2N(r, 0; \psi^{(r)}) + 2T(r, \xi) + \eta(2 + \tau)T(r, \xi) + T(r, L^n \psi^{(r)}) + T(r, L^n \psi) + N(r, 0; \xi^n) + N(r, 0; \phi^{(r)}) + S(r, \xi) + S(r, L) \]
\[ \leq (\eta(3\tau + 4) + 4)T(r, L) + (\eta(2\tau + 3) + 3)T(r, \xi) + T(r, L^n \psi^{(r)}) - T(r, L^n \psi) + S(r, \xi) + S(r, L) \tag{4.4} \]

Hence from (4.4) we have

\[ T(r, L^n \psi) \leq (\eta(3\tau + 4) + 4)T(r, L) + (\eta(2\tau + 3) + 3)T(r, \xi) + S(r, \xi) + S(r, L) \tag{4.5} \]

By lemma 4.10 we have from (4.5)

\[ (n - \lambda)T(r, L) \leq (\eta(3\tau + 4) + 4)T(r, L) + (\eta(2\tau + 3) + 3)T(r, \xi) + S(r, \xi) + S(r, L) \tag{4.6} \]
Similarly we have

\[(n - \lambda)T(r, \xi) \leq (\eta(3\tau + 4) + 4)T(r, \xi) + (\eta(2\tau + 3) + 3)T(r, L) + S(r, \xi) + S(r, L)\]  

(4.7)

From (4.6) and (4.7) we have

\[(n - (\lambda + \eta(5\tau + 7) + 7))(T(r, L) + T(r, \xi)) \leq S(r, \xi) + S(r, L)\]  

(4.8)

From (4.8) we get \(n \leq \lambda + \eta(5\tau + 7) + 7\). □

**Lemma 4.13.** Let \(L\) be a nonconstant \(L\)-function and \(\xi\) be a transcendental meromorphic function such that \(\rho_2(L) < 1\), \(\rho_2(\xi) < 1\) and \(\xi\), \(L\) share (\(\infty, 0\)), Also let \(L(z)^n[\psi(z)]^{(r)}\) and \(\xi(z)^n[\phi(z)]^{(r)}\) share \((P(z), 1)\) and, where \(P(z)\) is a polynomial of \(z\). If \(\Omega_{\Phi, \Psi} \neq 0\) then \(n \leq \lambda + \eta(5\tau + 9) + 7\).

**Proof.** Using lemma 4.2, lemma 4.3, lemma 4.7 and proceeding as lemma 4.12 we can prove this lemma. □

**Lemma 4.14.** Let \(L\) be a nonconstant \(L\)-function and \(\xi\) be a transcendental meromorphic function such that \(\rho_2(L) < 1\), \(\rho_2(\xi) < 1\) and \(\xi\), \(L\) share (\(\infty, 0\)), Also let \(L(z)^n[\psi(z)]^{(r)}\) and \(\xi(z)^n[\phi(z)]^{(r)}\) share \((P(z), l)\), where \(2 \leq l < \infty\) and \(P(z)\) is a polynomial of \(z\). If \(\Omega_{\Phi, \Psi} \neq 0\) then \(n \leq \lambda + \eta(2\tau + 4) + 4\).

**Proof.** Using lemma 4.2, lemma 4.3, lemma 4.6 and proceeding as lemma 4.12 we can prove this lemma. □

**Lemma 4.15.** Let \(L\) be a nonconstant \(L\)-function and \(\xi\) be a transcendental meromorphic function such that \(\rho_2(L) < 1\), \(\rho_2(\xi) < 1\) and \(\xi\), \(L\) share (\(\infty, 0\)), Also let \(L(z)^n[\psi(z)]^{(r)}\) and \(\xi(z)^n[\phi(z)]^{(r)}\) share \((P(z), 0)\), where \(P(z)\) is a polynomial of \(z\). If \(\Omega_{\Phi, \Psi} \equiv 0\) and \(n > \lambda + 2\eta(\tau + 1) + 2\) then either \(L(z)^n[\psi(z)]^{(r)}[\xi(z)^n[\phi(z)]^{(r)}\equiv P(z)^2\) or \(L(z)^n[\psi(z)]^{(r)} \equiv \xi(z)^n[\phi(z)]^{(r)}\).

**Proof.** Since \(\Omega_{\Phi, \Psi} \equiv 0\) therefore \((\Phi''/\Phi') - (2\Phi'/\Phi - 1) = 0\).

Integrating we have

\[
\frac{1}{\Phi - 1} \equiv \frac{A - B(\Psi - 1)}{\Psi - 1},
\]

(4.9)
where \( A(\neq 0) \) and \( B \) are constants.

Now we have to consider the following two cases

**Case 1** Let \( B = 0 \). Then from (4.9) we have

\[
\frac{1}{1 - \Phi} = \frac{A}{1 - \Psi}.
\] (4.10)

If possible let \( A \neq 1 \), then from (4.10) we have

\[
\overline{N}(r, 0; \Phi) = \overline{N}(r, 1 - A; \Psi).
\] (4.11)

By lemma 4.2, lemma 4.3 lemma 4.9 using second fundamental theorem we have

\[
T(r, L^n\psi^{(\tau)}) = T(r, \Psi) + S(r, L) \\
\leq \overline{N}(r, 0; \Psi) + \overline{N}(r, 1 - A; \Psi) + \overline{N}(r, \infty; \Psi) + S(r, \Psi) \\
\leq \overline{N}(r, 0; \Phi) + \overline{N}(r, 0; \Psi) + S(r, \Psi) \\
\leq \overline{N}(r, 0; \xi^n\phi^{(\tau)}) + \overline{N}(r, 0; L^n\psi^{(\tau)}) + S(r, L) \\
\leq \overline{N}(r, 0; \xi^n) + \overline{N}(r, 0; \phi^{(\tau)}) + \overline{N}(r, 0; L^n) + \overline{N}(r, 0; \psi^{(\tau)}) + S(r, L) \\
\leq T(r, L) + \tau \overline{N}(r, \infty; \psi) + N_{\tau+1}(r, 0; \psi) + T(r, \xi) + \tau \overline{N}(r, \infty; \phi) \\
+ N_{\tau+1}(r, 0; \phi) + S(r, \xi) + S(r, L) \\
\leq (\eta(\tau + 1) + 1)T(r, L) + (\eta(\tau + 1) + 1)T(r, \xi) \\
+ S(r, \xi) + S(r, L) 
\] (4.12)

Hence we get from (4.12)

\[
T(r, L^n\psi^{(\tau)}) \leq (1 + \eta(\tau + 1))T(r, L) + (1 + \eta(\tau + 1))T(r, \xi) \\
+ S(r, \xi) + S(r, L) 
\] (4.13)

Using lemma 4.11 we get from (4.13)

\[
(n - \lambda)T(r, L) \leq (1 + \eta(\tau + 1))T(r, L) + (1 + \eta(\tau + 1))T(r, \xi) \\
+ S(r, \xi) + S(r, L) 
\] (4.14)

Similarly we have

\[
(n - \lambda)T(r, \xi) \leq (1 + \eta(\tau + 1))T(r, \xi) + (1 + \eta(\tau + 1))T(r, L) \\
+ S(r, \xi) + S(r, L) 
\] (4.15)
From (4.14) and (4.15) we have
\[
(n - \lambda - (2\eta(\tau + 1) + 2))(T(r, L) + T(r, \xi)) \leq S(r, \xi) + S(r, L),
\]
which contradicts \( n > \lambda + 2\eta(\tau + 1) + 2 \).
Hence \( A = 1 \) and therefore we get from (4.10)
\[
L(z)^n[\psi(z)]^{(\tau)} \equiv \xi(z)^n[\phi(z)]^{(\tau)}
\]
\[\textbf{Case 2} \]
Let \( B \neq 0 \). If possible let \( A \neq -B \).
If \( B = 1 \), then from (4.9) we have
\[
\frac{1}{\Phi} = \frac{1}{A}(1 + A - \Psi)
\]
(4.16)
Using lemma 4.3 we have from (4.16)
\[
N(r, A + 1; \Psi) = N(r, \infty; \Phi) = S(r, L)
\]
Proceeding as Case 1 we arrive at a contradiction.
If \( B \neq 1 \), then from (4.9) we have
\[
\frac{1}{\Phi - (1 - \frac{1}{B})} = \frac{B^2}{A} \left( \frac{A + B}{B} - \Psi \right).
\]
Hence we get by lemma 4.3
\[
N(r, \frac{A + B}{B}; \Psi) = N(r, \infty; \Phi) = S(r, L)
\]
Proceeding as Case 1 we arrive at a contradiction.
Hence \( A = -B \).
If \( B = 1 \), then from (4.9) we have \( \Phi \Psi \equiv 1 \). Hence
\[
L(z)^n[\psi(z)]^{(\tau)}\xi(z)^n[\phi(z)]^{(\tau)} \equiv P(z)^2.
\]
(4.17)
If \( B \neq 1 \), then from (4.9) we have \( \frac{1}{\Phi} = \frac{-B\Psi}{(1-B)\Psi - 1} \).
Hence \( N(r, 0; \Phi) = N(r, \frac{1}{1-B}; \Psi) \).
Proceeding as Case 1 we arrive at a contradiction.
5. PROOF OF THE MAIN RESULT

Proof of Theorem 3.1

Since $\xi, L$ share $(\infty, 0)$ and $\xi(z)^n[\phi(z)]^{(\tau)}, L(z)^n[\psi(z)]^{(\tau)}$ share $(P(z), 0)$ therefore $\Phi$ and $\Psi$ share $(1, l)$ except zeros of $P(z)$ and share $(\infty, 0)$.

By lemma 4.1 it is clear that $L$ is a transcendental meromorphic function.

Since $L$ and $\xi$ are transcendental meromorphic functions therefore $P$ is a small function of $L$ and $\xi$.

Now we have to consider the following two cases

**Case 1** Let $\Omega_{\Phi, \Psi} \not= 0$.

By lemma 4.12, lemma 4.13 and lemma 4.14 we arrive at a contradiction.

**Case 2** Let $\Omega_{\Phi, \Psi} \equiv 0$.

Hence by lemma 4.15 one of the following holds

(i) $L(z)^n[\prod_{j=1}^{\eta} (z + \omega_j)^{\mu_j}]^{(\tau)} \xi(z)^n[\prod_{j=1}^{\eta} (z + \omega_j)^{\mu_j}]^{(\tau)} \equiv P(z)^2$;

(ii) $L(z)^n[\prod_{j=1}^{\eta} (z + \omega_j)^{\mu_j}]^{(\tau)} \equiv \xi(z)^n[\prod_{j=1}^{\eta} (z + \omega_j)^{\mu_j}]^{(\tau)}$. If

$$L(z)^n[\prod_{j=1}^{\eta} (z + \omega_j)^{\mu_j}]^{(\tau)} \xi(z)^n[\prod_{j=1}^{\eta} (z + \omega_j)^{\mu_j}]^{(\tau)} \equiv P(z)^2,$$  \hspace{1cm} (5.1)

then by lemma 4.4, lemma 4.5 and lemma 4.9 we get from (5.1)

$$n[N(r, \infty; L) + N(r, \infty; \xi)] \leq N(r, 0; \phi^{(\tau)}) + N(r, 0; \psi^{(\tau)}) \leq N(r, 0; \phi) + \tau N(r, \infty; \phi) + N(r, 0; \psi) + \tau N(r, \infty; \psi) + S(r, L) + S(r, \xi) \leq \lambda[N(r, 0; L) + N(r, 0; \xi)] + \tau \eta[N(r, \infty; L) + N(r, \infty; \xi)] + S(r, L) + S(r, \xi).$$ \hspace{1cm} (5.2)

Similarly we have

$$n[N(r, 0; L) + N(r, 0; \xi)] \leq (\lambda + \tau \eta)[N(r, \infty; L) + N(r, \infty; \xi)] + S(r, L) + S(r, \xi).$$ \hspace{1cm} (5.3)

By lemma 4.2 and lemma 4.3 we get from (5.2) and (5.3)

$$(n - \lambda)[N(r, 0; L) + N(r, 0; \xi)] \leq S(r, L) + S(r, \xi).$$ \hspace{1cm} (5.4)

Since $n > \lambda$ therefore from (5.4) we can conclude that $L$ and $\xi$ have finitely many zeros. If possible let $z_1$ be a pole of $\xi$ of multiplicity $m$. 
Since $L$ and $\xi$ share $(\infty, 0)$ therefore without loss of generality we may assume $z_1$ is a pole of order $k$. If $z_1$ is a zero of $\phi^{(r)}$ and $\psi^{(r)}$ then we get from (5.1)

$$n(m + k) \leq \sum_{j=1}^{\eta} a_j \mu_j + \sum_{j=1}^{\eta} b_j \mu_j - 2\tau \leq c_1 + c_2.$$  \hspace{1cm} (5.5)

From (5.5) we have $n \leq \max\{c_1, c_2\}$, which contradicts $n > \max\{\lambda, c_1, c_2\}$. Similarly we get a contradiction if we assume that $z_1$ is a zero of $\phi^{(r)}$ but not a zero of $\psi^{(r)}$ and vice versa.

Hence $\xi$ has no poles.

Since $\xi$ and $L$ share $(\infty, 0)$ therefore $L$ also has no pole. Hence $\xi$ and $L$ are entire functions and so $\phi^{(r)}$ and $\psi^{(r)}$ are entire functions.

We can also deduce from (5.1) that $\xi$ and $L$ have no zeros.

Hence $L = e^{a(z)}$ and $\xi = e^{b(z)}$, where $a(z)$ and $b(z)$ are entire functions.

This completes the proof of the theorem.

REFERENCES


