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Bounded Weak Solutions for Hilfer Fractional Differential Equations on the Half Line

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Abstract

In this paper, we discuss the existence of bounded weak solutions for a class of Hilfer differential equations on an unbounded domain. The main results are proved with the aid of Mönch's fixed point theorem, measure of weak noncompactness and the diagonalization method.

Keywords: Hilfer fractional derivative; Pettis–Riemann–Liouville fractional integral; unbounded domain; weak solution; diagonalization process

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1. INTRODUCTION

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences [18, 27]. For some fundamental results in the theory of fractional calculus and fractional differential equations, we refer the reader to the monographs by Abbas *et al.* [1, 2, 3], Samko *et al.* [25], Kilbas *et al.* [20] and Zhou [29]. Recently, considerable attention has been given to the existence of solutions for initial and boundary value problems of fractional differential equations involving Hilfer fractional derivative, for instance, see [18, 19, 28].

The idea of measure of weak noncompactness was introduced by De Blasi [14]. The concept of strong measure of noncompactness, introduced by Banaŝ and Goebel [7], was further developed and used by several authors, for example, see Akhmerov *et al.* [4], Alvàrez [5], Benchohra *et al.* [12], Guo *et al.* [17], and the references therein. In [12, 23], the authors obtained some existence results for fractional and ordinary differential equations respectively by applying the techniques of the measure of noncompactness. For some recent works concerning the application of this technique, see [3, 9, 10], and the references therein.

In [6, 8, 11], the diagonalization method was applied to prove the existence of bounded solutions for several classes of fractional differential equations on the half line.

In this paper, we investigate the existence of bounded weak solutions for the following problem involving the Hilfer fractional derivative:

$$\begin{cases} (D_0^{\alpha,\beta}u)(t) = f(t,u(t));\ t \in \mathbb{R}_+ := [0,\infty), \\ \\ (I_0^{1-\gamma}u)(t)|_{t=0} = \phi,\ u \text{ is bounded on } \mathbb{R}_+, \end{cases} \tag{1}$$

where $\alpha \in (0,1),\ \beta \in [0,1],\ \gamma = \alpha+\beta-\alpha\beta,\ \phi \in E,\ f : \mathbb{R}_+ \times E \to E$ is a given continuous function, E is a real (or complex) reflexive Banach space with norm $\|\cdot\|_E$ and dual E^*, such that E is the dual of a weakly compactly generated Banach space X, $I_0^{1-\gamma}$ is the left-sided mixed Riemann–Liouville integral of order $1-\gamma$, and $D_0^{\alpha,\beta}$ is Hilfer fractional derivative operator of order α and type β.

Here we emphasize that the present work initiates the application of the diagonalization method in the context of Hilfer fractional differential equations under the weak topology.

2. PRELIMINARIES

Let $I_n := [0, n],\ n \in \mathbb{N}^*$ and $C_n := C(I_n)$ be the Banach space of all continuous functions v from I_n into E with the supremum (uniform) norm

$$\|v\|_n := \sup_{t \in I_n} \|v(t)\|_E.$$

As usual, $AC(I_n)$ denotes the space of absolutely continuous functions from I_n into E. We denote by $AC^1(I_n)$ the space defined by

$$AC^1(I_n) := \{w : I \to E : \frac{d}{dt} w(t) \in AC(I_n)\}.$$

By $C_\gamma(I_n)$ and $C^1_\gamma(I_n)$, we denote the weighted spaces of continuous functions defined by

$$C_\gamma(I_n) = \{w : (0, T] \to E : t^{1-\gamma} w(t) \in C(I_n)\},$$

with the norm

$$\|w\|_{C_\gamma} := \sup_{t \in I_n} \|t^{1-\gamma} w(t)\|_E,$$

and

$$C^1_\gamma(I_n) = \{w \in C : \frac{dw}{dt} \in C_\gamma\},$$

with the norm

$$\|w\|_{C^1_\gamma} := \|w\|_\infty + \|w'\|_{C_\gamma}.$$

In the following we denote $\|w\|_{C_\gamma}$ by $\|w\|_C$. Let $(E, w) = (E, \sigma(E, E^*))$ be the Banach space E with its weak topology.

Definition 2.1. *A Banach space X is called weakly compactly generated (WCG, for short) if it contains a weakly compact set whose linear span is dense in X.*

Definition 2.2. *A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any (u_n) in E with $u_n \to u$ in (E, w) then $h(u_n) \to h(u)$ in (E, w)).*

Definition 2.3. *[24] The function $u : I \to E$ is said to be Pettis integrable on I if and only if there is an element $u_J \in E$ corresponding to each $J \subset I$ such that $\phi(u_J) = \int_J \phi(u(s))ds$ for all $\phi \in E^*$, where the integral on the right hand side is assumed to exist in the sense of Lebesgue, (by definition, $u_J = \int_J u(s)ds$).*

Let $P(I_n, E)$ be the space of all $E-$valued Pettis integrable functions on I_n, and $L^1(I_n, \mathbb{R})$ be the Banach space of Lebesgue integrable functions $u : I_n \to \mathbb{R}$. Define the class $P_1(I_n, E)$ by

$$P_1(I_n, E) = \{u \in P(I_n, E) : \varphi(u) \in L^1(I_n, \mathbb{R});\ for\ every\ \varphi \in E^*\}.$$

The space $P_1(I_n, E)$ is normed by

$$\|u\|_{P_1} = \sup_{\varphi\in E^*,\ \|\varphi\|\leq 1} \int_0^n |\varphi(u(x))| d\lambda x,$$

where λ stands for a Lebesgue measure on I_n.

The following result is due to Pettis (see [[24], Theorem 3.4 and Corollary 3.41]).

Proposition 2.4. *[24] If $u \in P_1(I_n, E)$ and h is a measurable and essentially bounded real-valued function, then $uh \in P_1(I_n, E)$.*

For all that follows, the symbol "$\int$" denotes the Pettis integral.

Theorem 2.5. *[26] A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

Theorem 2.6. *[21] Let D be a weakly compact subset of $C(I_n, E)$. Then $D(t)$ is weakly compact subset of E for each $t \in I$.*

Now, we give some results and properties of fractional calculus.

Definition 2.7. *[2, 20, 25] The left-sided mixed Riemann–Liouville integral of order $r > 0$ of a function $w \in L^1(I_n)$ is defined by*

$$(I_\theta^r w)(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} w(s) ds; \textit{ for a.e. } t \in I_n,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt;\ \xi > 0.$$

For all $r, r_1, r_2 > 0$ and each $w \in C(I_n)$, notice that $I_0^r w \in C(I_n)$, and

$$(I_0^{r_1} I_0^{r_2} w)(t) = (I_0^{r_1+r_2} w)(t);\ \text{for a.e. } t \in I_n.$$

Definition 2.8. *([20]) The Riemann–Liouville fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I_n)$ is defined by*

$$\begin{aligned}(D_0^r w)(t) &= \left(\frac{d}{dt} I_0^{1-r} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} w(s) ds; \textit{ for a.e. } t \in I_n.\end{aligned}$$

Let $r \in (0,1],\ \gamma \in [0,1)$ and $w \in C_{1-\gamma}(I_n)$. Then the following integral becomes the left inverse operator:

$$(D_0^r I_0^r w)(t) = w(t);\ \text{for all}\ t \in (0,n].$$

Moreover, if $I_0^{1-r} w \in C^1_{1-\gamma}(I_n)$, then the following composition is proved in [25]:

$$(I_0^r D_0^r w)(t) = w(t) - \frac{(I_0^{1-r} w)(0^+)}{\Gamma(r)} t^{r-1};\ \text{for all}\ t \in (0,n].$$

Definition 2.9. *[20] The Caputo fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I_n)$ is defined by*

$$\begin{aligned}({}^cD_0^r w)(t) &= \left(I_0^{1-r}\frac{d}{dt}w\right)(t)\\ &= \frac{1}{\Gamma(1-r)}\int_0^t (t-s)^{-r}\frac{d}{ds}w(s)ds;\ \textit{for a.e.}\ t \in I_n.\end{aligned}$$

In [18], Hilfer studied applications of a generalized fractional operator having the Riemann–Liouville and the Caputo derivatives as specific cases (see also [19, 28].

Definition 2.10. *(Hilfer derivative). Let $\alpha \in (0,1),\ \beta \in [0,1],\ w \in L^1(I_n),\ I_0^{(1-\alpha)(1-\beta)} w \in AC(I_n)$. The Hilfer fractional derivative of order α and type β of w is defined as*

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)}\frac{d}{dt} I_0^{(1-\alpha)(1-\beta)} w\right)(t);\ \textit{for a.e.}\ t \in I_n. \tag{2}$$

Now we enlist some properties satisfied by the Hilfer fractional derivative ([15, 20]).

Properties. Let $\alpha \in (0,1),\ \beta \in [0,1],\ \gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(I_n)$.
1. The operator $(D_0^{\alpha,\beta} w)(t)$ can be written as

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)}\frac{d}{dt} I_0^{1-\gamma} w\right)(t) = \left(I_0^{\beta(1-\alpha)} D_0^{\gamma} w\right)(t);\ for\ a.e.\ t \in I_n.$$

Moreover, the parameter γ satisfies

$$\gamma \in (0,1],\ \gamma \geq \alpha,\ \gamma > \beta,\ 1-\gamma < 1-\beta(1-\alpha).$$

2. For $\beta = 0$ and $\beta = 1$, the Hilfer fractional derivative (2) respectively reduces to the Riemann–Liouville derivative and the Caputo derivative:

$$D_0^{\alpha,0} = D_0^{\alpha},\ and\ D_0^{\alpha,1} = {}^cD_0^{\alpha}.$$

3. If $D_0^{\beta(1-\alpha)}w$ exists and it is in $L^1(I_n)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = (I_0^{\beta(1-\alpha)} D_0^{\beta(1-\alpha)} w)(t);\ for\ a.e.\ t \in I_n.$$

Furthermore, if $w \in C_\gamma(I_n)$ and $I_0^{1-\beta(1-\alpha)} w \in C^1_\gamma(I_n)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = w(t);\ for\ a.e.\ t \in I_n.$$

4. If $D_0^{\gamma} w$ exists and in $L^1(I_n)$, then

$$(I_0^{\alpha} D_0^{\alpha,\beta} w)(t) = (I_0^{\gamma} D_0^{\gamma} w)(t) = w(t) - \frac{I_0^{1-\gamma} w(0^+)}{\Gamma(\gamma)} t^{\gamma-1};\ for\ a.e.\ t \in I_n.$$

Lemma 2.11. *([15]) Let $h \in C_\gamma(I_n)$. Then the linear problem*

$$\begin{cases} (D_0^{\alpha,\beta} u)(t) = h(t);\ t \in I_n, \\ \\ (I_0^{1-\gamma} u)(t)|_{t=0} = \phi,\ \gamma = \alpha + \beta - \alpha\beta, \end{cases}$$

has a unique solution $u \in L^1(I_n, E)$ given by

$$u(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^{\alpha} h)(t).$$

In view of the above lemma, we can state the following result for the problem (1).

Lemma 2.12. *Let $f : I_n \times E \to E$ be such that $f(\cdot, u(\cdot)) \in C_\gamma(I_n)$ for any $u \in C_\gamma(I_n)$. Then problem (1) is equivalent to the problem of the solutions of the integral equation*

$$u(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^{\alpha} f(\cdot, u(\cdot)))(t),\ \gamma = \alpha + \beta - \alpha\beta.$$

Remark 2.13. *Let $g \in P_1([I_n, E)$. For every $\varphi \in E^*$, we have*

$$\varphi(I_0^{\alpha} g)(t) = (I_0^{\alpha} \varphi g)(t);\ for\ a.e.\ t \in I_n.$$

Definition 2.14. *[14] Let Ω_E be a bounded subset of a Banach space E and B_1 be the unit ball in E. The De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \to [0, \infty)$ defined by*

$$\beta(X) = \inf\{\epsilon > 0 : \textit{there exists a weakly compact } \Omega \subset E \text{ such that } X \subset \epsilon B_1 + \Omega\}.$$

The De Blasi measure of weak noncompactness satisfies the following properties:

(a) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$,

(b) $\beta(A) = 0 \Leftrightarrow A$ is relatively weakly compact,

(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$,

(d) $\beta(\overline{A}^{\omega}) = \beta(A)$, ($\overline{A}^{\omega}$ denotes the weak closure of A),

(e) $\beta(A + B) \leq \beta(A) + \beta(B)$,

(f) $\beta(\lambda A) = |\lambda|\beta(A)$,

(g) $\beta(conv(A)) = \beta(A)$,

(h) $\beta(\cup_{|\lambda| \leq h} \lambda A) = h\beta(A)$.

The next result follows directly from the Hahn–Banach theorem.

Proposition 2.15. *Let E be a normed space, and $x_0 \in E$ with $x_0 \neq 0$. Then, there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

Let us define a set V of functions $v : I_n \to E$ as follows:

$$V(t) = \{v(t) : v \in V\};\ t \in I_n,$$

and

$$V(I) = \{v(t) : v \in V,\ t \in I_n\}.$$

Lemma 2.16. *[17] Let $H \subset C$ be a bounded and equicontinuous subset. Then the function $t \to \beta(H(t))$ is continuous on I_n, and*

$$\beta_C(H) = \max_{t \in I_n} \beta(H(t)),$$

and

$$\beta\left(\int_I u(s)ds\right) \leq \int_{I_n} \beta(H(s))ds,$$

where $H(s) = \{u(s) : u \in H,\ s \in I\}$, and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of $C(I_n)$.

For our purpose we will need the following fixed point theorem:

Theorem 2.17. *[22] Let Q be a nonempty, closed, convex and equicontinuous subset of a metrizable locally convex vector space $C(I,E)$ such that $0 \in Q$. Suppose $T : Q \to Q$ is weakly-sequentially continuous. If*

$$\overline{V} = \overline{conv}(\{0\} \cup T(V)) \Rightarrow V \text{ is relatively weakly compact}, \tag{3}$$

for every subset $V \subset Q$, then the operator T has a fixed point.

3. EXISTENCE OF BOUNDED WEAK SOLUTIONS

Let us begin this section by defining a weak solution to the problem (1).

Definition 3.1. *A measurable and bounded function $u \in C_\gamma$ is said to be a weak solution to the problem (1) if it satisfies the condition $(I_0^{1-\gamma}u)(0^+) = \phi$, and the equation $(D_0^{\alpha,\beta}u)(t) = f(t,u(t))$ on $\mathbb{R}_+$.*

The following hypotheses will be used in the sequel.

(H_1) For a.e. $t \in I_n$, the function $v \to f(t,v)$ is weakly sequentially continuous.

(H_2) For each $v \in E$, the function $t \to f(t,v)$ is Pettis integrable on I_n.

(H_3) There exists $p_n \in C(I_n, [0,\infty))$ such that for all $\varphi \in E^*$, we have

$$|\varphi(f(t,u))| \leq p_n(t), \ for \ a.e. \ t \in I_n, \ and \ each \ u \in E.$$

(H_4) For each bounded and measurable set $B \subset E$ and for each $t \in I_n$, we have

$$\beta(f(t,B)) \leq t^{1-r} p_n(t) \beta(B).$$

Theorem 3.2. *Assume that the hypotheses $(H_1) - (H_4)$ and the following condition hold:*

$$L_n := \frac{p_n^* n^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} < 1, \tag{4}$$

where $p_n^* = \sup_{t \in I_n} p_n(t)$. Then the problem (1) has at least one bounded weak solution on $\mathbb{R}_+$.

Proof. We complete the proof in two parts. Fix $n \in \mathbb{N}$ and consider the problem

$$\begin{cases} (D_0^{\alpha,\beta}u)(t) = f(t,u(t)); \ t \in I_n, \\ \\ (I_0^{1-\gamma}u)(t)|_{t=0} = \phi. \end{cases} \tag{5}$$

Part1. We begin by showing that (5) has a solution $u_n \in C_\gamma(I_n)$ with $\|u_n\|_{C_\gamma} \le R_n$ for each $t \in I_n,$ where

$$R_n \ge \frac{p_n^* n^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}.$$

Consider the operator $N : C_\gamma(I_n) \to C_\gamma(I_n)$ defined by

$$(Nu)(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + \int_0^t (t-s)^{\alpha-1} \frac{f(s, u(s))}{\Gamma(\alpha)} ds. \tag{6}$$

By the given hypotheses, for each $u \in C_\gamma(I_n)$, the function $t \mapsto (t-s)^{\alpha-1} f(s, u(s))$, $for\ a.e.\ t \in I_n,$ is Pettis integrable. Thus, the operator N is well defined.

Consider the set

$$Q_n = \Big\{ u \in C_\gamma(I_n) : \|u\|_n \le R_n \text{ and } \|t_2^{1-\gamma} u(t_2) - t_1^{1-\gamma} u(t_1)\|_E$$
$$\le \frac{p_n^* n^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} (t_2 - t_1)^\alpha + \frac{p_n^*}{\Gamma(\alpha)} \int_0^{t_1} |t_2^{1-\gamma}(t_2 - s)^{\alpha-1} - t_1^{1-\gamma}(t_1 - s)^{\alpha-1}| ds,\ t_1,\ t_2 \in I_n \Big\},$$

which is obviously closed, convex and equicontinuous. We shall show that the operator N satisfies all the assumptions of Theorem 2.17. The proof will be given in several steps.

Step 1. *N maps Q_n into itself.*
Let $u \in Q_n,\ t \in I_n$ and assume that $(Nu)(t) \ne 0$. Then there exists $\varphi \in E^*$ such that $\|t^{1-\gamma}(Nu)(t)\|_E = |\varphi(t^{1-\gamma}(Nu)(t))|$. Thus

$$\|t^{1-\gamma}(Nu)(t)\|_E = \left| \varphi\left(\frac{\phi}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds \right) \right|.$$

Then

$$\begin{aligned}
\|t^{1-\gamma}(Nu)(t)\|_E &\le \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\varphi(f(s, u(s)))| ds \\
&\le \frac{p_n^* n^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&\le \frac{p_n^* n^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} \\
&\le R_n.
\end{aligned}$$

Next, let $t_1, t_2 \in I$ such that $t_1 < t_2$ and let $u \in Q_n$, with

$$t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1) \ne 0.$$

Then there exists $\varphi \in E^*$ such that

$$\|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\|_E = |\varphi(t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1))|,$$

and $\|\varphi\| = 1$. In consequence,

$$\|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\|_E = |\varphi(t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1))|$$
$$\leq \left|\varphi\left(t_2^{1-\gamma}\int_0^{t_2}(t_2-s)^{\alpha-1}\frac{f(s,u(s))}{\Gamma(\alpha)}ds - t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}\frac{f(s,u(s))}{\Gamma(\alpha)}ds\right)\right|.$$

and

$$\|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\|_E \leq t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\frac{|\varphi(f(s,u(s)))|}{\Gamma(\alpha)}ds$$
$$+\int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|\frac{|\varphi(f(s,u(s)))|}{\Gamma(\alpha)}ds$$
$$\leq t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\frac{p_n(s)}{\Gamma(\alpha)}ds$$
$$+\int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|\frac{p_n(s)}{\Gamma(\alpha)}ds.$$

Thus, we get

$$\|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\|_E$$
$$\leq \frac{p_n^* n^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}(t_2-t_1)^\alpha + \frac{p_n^*}{\Gamma(\alpha)}\int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|ds.$$

Hence $N(Q_n) \subset Q_n$.

Step 2. *N is weakly-sequentially continuous.*
Let (u_m) be a sequence in Q_n and let $(u_m(t)) \to u(t)$ in (E,ω) for each $t \in I_n$. Fix $t \in I_n$, since f satisfies the assumption (H_1), we have that $f(t,u_m(t))$ converges weakly to $f(t,u(t))$. Hence it follows by the Lebesgue dominated convergence theorem for Pettis integral (see [16]) that $(Nu_m)(t)$ converges weakly to $(Nu)(t)$ in (E,ω), for each $t \in I_n$. Thus, $N(u_m) \to N(u)$. Hence, $N : Q_n \to Q_n$ is weakly-sequentially continuous.

Step 3. *The implication (3) holds.*
Let V be a subset of Q_n such that $\overline{V} = \overline{conv}(N(V) \cup \{0\})$. Obviously

$$V(t) \subset \overline{conv}(NV)(t)) \cup \{0\}),\ t \in I_n.$$

Further, as V is bounded and equicontinuous, by Lemma 3 in [13] the function $t \to v(t) = \beta(V(t))$ is continuous on I_n. From (H_3), (H_4), Lemma 2.16 and the properties of the measure β, for any $t \in I_n$, we have

$$t^{1-\gamma}v(t) \leq \beta(t^{1-\gamma}(NV)(t) \cup \{0\})$$

$$\begin{aligned}
&\leq \beta(t^{1-\gamma}(NV)(t)) \\
&\leq \frac{T^{1-\gamma}}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}p_n(s)\beta(V(s))ds \\
&\leq \frac{n^{1-\gamma}}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}s^{1-\gamma}p_n(s)v(s)ds \\
&\leq \frac{p_n^* n^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}\|v\|_C.
\end{aligned}$$

Thus

$$\|v\|_C \leq L_n\|v\|_C.$$

From (4), we get $\|v\|_C = 0$, that is, $v(t) = \beta(V(t)) = 0$, for each $t \in I_n$ and hence V is weakly relatively compact in $C_\gamma(I_n)$ by Theorem 2 in [21]. Applying now Theorem 2.17, we conclude that N has a fixed point $u_n \in C_\gamma(I_n)$ which is a weak solution to the problem (5) on I_n with $\|u_n\|_{C_\gamma} \leq R_n$ for each $t \in I_n$.

Part 2. For $k \in \mathbb{N}$, we use the following diagonalization process:

$$\begin{cases} w_k(t) = u_{n_k}(t); \ t \in [0, n_k], \\ w_k(t) = u_{n_k}(n_k); \ t \in [n_k, \infty). \end{cases}$$

Here $(n_k)_{k\in\mathbb{N}^*}$ is a sequence of numbers satisfying

$$0 < n_1 < n_2 < \ldots n_k < \ldots \uparrow \infty.$$

Let $S = \{w_k\}_{k=1}^{\infty}$. Notice that

$$|w_{n_k}(t)| \leq R_n : \ for \, t \in [0, n_1], \ k \in \mathbb{N}.$$

For $k \in \mathbb{N}$ and $t \in [0, n_1]$, we have

$$w_{n_k}(t) = \frac{\phi}{\Gamma(\gamma)}t^{\gamma-1} + \int_0^{n_1}(t-s)^{\alpha-1}\frac{f(s, w_{n_k}(s))}{\Gamma(\alpha)}ds.$$

Thus, for $k \in \mathbb{N}$ and $t, x \in [0, n_1]$, we have

$$|t^{1-\gamma}w_{n_k}(t) - x^{1-\gamma}w_{n_k}(x)| \leq \int_0^{n_1} n_1^{1-\gamma}|(t-s)^{\alpha-1} - (x-s)^{\alpha-1}|\frac{f(s, w_{n_k}(s))}{\Gamma(\alpha)}ds.$$

Hence

$$|t^{1-\gamma}w_{n_k}(t) - t^{1-\gamma}w_{n_k}(x)| \leq \frac{p_1^* n_1^{1-\gamma}}{\Gamma(\alpha)}\int_0^{n_1} |(t-s)^{\alpha-1} - (x-s)^{\alpha-1}|ds.$$

So the Arzelà–Ascoli theorem guarantees that there is a subsequence P_1^* of $\mathbb{N}$ and a function $z_1 \in C([0, n_1], \mathbb{R})$ with $w_{n_k} \to z_1$ as $k \to \infty$ in $C([0, n_1], \mathbb{R})$ through P_1^*. Let

$P_1 = P_1^* - \{1\}$.
Notice that

$$|w_{n_k}(t)| \leq R_n : \ for\ t \in [0, n_2],\ k \in \mathbb{N}.$$

Also, for $k \in \mathbb{N}$ and $t, x \in [0, n_2]$, we have

$$|t^{1-\gamma} w_{n_k}(t) - t^{1-\gamma} w_{n_k}(x)| \leq \frac{p_2^* n_2^{1-\gamma}}{\Gamma(\alpha)} \int_0^{n_2} |(t-s)^{\alpha-1} - (x-s)^{\alpha-1}| ds.$$

The Arzelà–Ascoli Theorem guarantees that there is a subsequence P_2^* of $\mathbb{N}_1$ and a function $z_2 \in C([0, n_2], E)$ with $w_{n_k} \to z_2$ as $k \to \infty$ in $C([0, n_2], E)$ through P_2^*. Note that $z_1 = z_2$ on $[0, n_1]$ since $P_2^* \subset P_1$. Let $P_2 = P_2^* - \{2\}$. For $m = 3, 4, \ldots$, proceed inductively to obtain a subsequence P_m^* of P_{m-1} and a function $z_m \in C([0, n_m], E)$ with $w_{n_k} \to z_m$ as $k \to \infty$ in $C([0, n_m], E)$ through P_m^*. Let $P_m = P_m^* - \{m\}$.
Fix $t \in (0, \infty)$ and let $m \in \mathbb{N}$ with $t \leq n_m$ and define $u(t) = z_m(t)$. Then $u \in C((0, \infty), E)$ and $|u(t)| \leq R_n$ for $t \in [0, \infty)$.
Again, fix $t \in (0, \infty)$ and let $m \in \mathbb{N}$ with $t \leq n_m$. Then, for $n \in P_m$, we have

$$u_{n_k}(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + \int_0^{n_m} (t-s)^{\alpha-1} \frac{f(s, u_{n_k}(s))}{\Gamma(\alpha)} ds.$$

Taking $n_k \to \infty$ through P_m, we obtain

$$z_m(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + \int_0^{n_m} (t-s)^{\alpha-1} \frac{f(s, z_m(s))}{\Gamma(\alpha)} ds.$$

We can repeat this method for each $t \in [0, n_m]$ and for each $m \in \mathbb{N}$. Thus

$$(D_0^{\alpha,\beta} u)(t) = f(t, u(t));\ for\ t \in [0, n_m],$$

for each $m \in \mathbb{N}$. Hence the constructed function u is a solution to the problem (1).

4. AN EXAMPLE

Let

$$l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{m=1}^{\infty} |u_m| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_{l^1} = \sum_{m=1}^{\infty} |u_m|.$$

Consider the following problem of Hilfer fractional differential equation

$$\begin{cases} (D_0^{\frac{1}{2},\frac{1}{2}} u_m)(t) = f_m(t, u(t));\ t \in \mathbb{R}_+, \\ (I_0^{\frac{1}{4}} u)(t)|_{t=0} = (2^{-1}, 2^{-2}, \ldots, 2^{-m}, \ldots),\ u \text{ is bounded on } \mathbb{R}_+, \end{cases} \tag{7}$$

where

$$f_m(t,u(t)) = \frac{c_n(2^{-n}+u_m(t))}{e^{t+4}(1+\|u(t)\|_{l^1})};\ u \in l^1,$$

for each $t \in [0,n];\ n \in \mathbb{N}^*$, with

$$u = (u_1, u_2, \ldots, u_m, \ldots),\ and\ c_n := \frac{e^4}{8\, n^{\frac{3}{4}}}\Gamma\left(\frac{1}{2}\right).$$

Set

$$f = (f_1, f_2, \ldots, f_m, \ldots),$$

$\alpha = \beta = \frac{1}{2}$, then $\gamma = \frac{3}{4}$.
For each $u \in l^1$ and $t \in \mathbb{R}_+$, we have

$$\|f(t,u(t))\|_{l^1} \le \frac{c_n}{e^{t+4}}.$$

Hence, the hypothesis (H_3) is satisfied with $p_n^* = c_n e^{-4}$. Also the condition (4) holds true as

$$\frac{p_n^* n^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} = \frac{n^{3/4}e^{-4}}{\Gamma(3/2)}c_n = \frac{n^{3/4}e^{-4}}{\Gamma(3/2)}\left(\frac{e^4\Gamma(1/2)}{8\, n^{\frac{3}{4}}}\right) = \frac{1}{4} < 1.$$

A simple computation shows that all conditions of Theorem 3.2 are satisfied. Hence it follows by the conclusion of Theorem 3.2 that the problem (7) has at least one bounded weak solution on $\mathbb{R}_+$.

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