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On Hyers-Ulam and Hyers-Ulam-Rassias stability of Second Order Linear Dynamic Equations on Time Scales

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Abstract

In this paper, we investigate new sufficient conditions for Hyers-Ulam and Hyers-Ulam-Rassias stability of second order linear dynamic equations on time scales of the form

$$\psi^{\Delta^2}(t) + p(t)\psi(t) - f(t) = 0, \quad t \in [a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T},$$

where $\mathbb{T}$ is a time scale and f is rd-continuous from $[a,b]_{\mathbb{T}}$ to a Banach space X. Our results depend on creating an equivalent integral equation and using the fixed point theorem.

Keywords: time scales, second order linear dynamic equations on time scales, Hyers-Ulam stability and Hyers-Ulam-Rassias stability.

1. PRELIMINARIES AND INTRODUCTION

In 1940, Ulam presented the following problem related to the stability of functional equations: "give conditions in order for a linear mapping near an approximately linear mapping to exist". See [21]. The case of approximately additive mappings was solved by Hyers [7] who proved that the Cauchy equation is stable in Banach spaces.

Since then, this type of stability founded by Ulam and Hyers, famed for Hyers-Ulam stability. Recently, there has been hundreds papers appeared concerning Hyers-Ulam stability due to its applications in control theory and numerical analysis etc. In 1978, Rassias [15] extended Hyers–Ulam stability concept and called it Hyers–Ulam–Rassias stability. For more details, we refer the reader to the monograph of Jung [9].

In 1998, Alsina and Ger [5] were first authors who investigated the Hyers-Ulam stability of differential equations. This result has been generalized by Miura,Takahasi and Choda [10], by Miura [11], and by Miura, Takahasi and Miyajima [12], [13]. Popa proved tha Hyers-Ulam stability of linear recurrence with constant coefficients [14]. Many articles, dealing with Hyers-Ulam stability, were edited by Rassias [16]. Wang, Zhou and Sun introduced the Heyrs-Ulam stability of linear differential equations of first order [22]. In 2012 Anderson, Gates and Heuer [1] extended the work of Li and Shen [18, 19] to prove the Hyers-Ulam stability of the scalar second order linear non-homogeneous dynamic equation on bounded time scales. They obtained their results via a related Riccati dynamic equation. Also in 2012 András and Mészáros studied the Ulam-Hyers stability of some linear and nonlinear dynamic equations and integral equations on time scales based on the theory of Picard operators [2]. Hamza and Yassen extended the work of Douglas, Gates and Heuer, and investigated Hyers-Ulam stability of abstract second order linear dynamic equations on unbounded time scales [6]. In 2017, Shen established Ulam stability of first order linear dynamic equations and its adjoint equation on time scales by using the integrating factor method [20]. Recently there has been a great interest in studying stability of dynamic equations on time scales.

In this paper, we investigate new sufficient conditions for Hyers-Ulam and Hyers-Ulam-Rassias stability of second order linear dynamic equations on time scales of the form

$$\psi^{\Delta^2}(t) + p(t)\psi(t) - f(t) = 0, \quad t \in [a, b]_{\mathbb{T}} \tag{1.1}$$

where $\mathbb{T}$ is a time scale, $p \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ and $f \in C_{rd}([a, b]_{\mathbb{T}}, X)$. Here, $[a, b]_{\mathbb{T}}$ is the time scale

$$[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}.$$

Our results depend basically on finding an equivalent integral equation to equation (1.1). The main result of the paper is that a sufficient condition for equation (1.1) to have Hyers-Ulam stability is the existence of a unique solution ψ satisfying the initial conditions $\psi^{\Delta^i}(a) = a_i, i = 0, 1$ for any initial values $a_0, a_1 \in X$.

For the terminology and notations used here, we refer the reader to the very interesting monographs of Bohner and Peterson [3] and [4]. We start the paper by introducing some of the basic definitions and notations of the calculus of time scales.

Definition 1.1. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$.

Definition 1.2. Let $\mathbb{T}$ be a time scale. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

In this definition we put $\inf \emptyset = \sup \mathbb{T}$.

Definition 1.3. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

From now on, X denotes a Banach space with a norm $\|.\|$.

Definition 1.4. (1) For a function $f : \mathbb{T} \to X$, $f^\sigma(t)$ is understood to mean $f(\sigma(t))$.

(2) A function $f : \mathbb{T} \to X$ is said to be right-dense continuous or rd-continuous provided f is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points t in $\mathbb{T}$.
The set of all rd-continuous functions $f : \mathbb{T} \to X$ will be denoted by $C_{rd}(\mathbb{T}, X)$.

(3) Assume $f : \mathbb{T} \to X$, and let $t \in \mathbb{T}^k$. The delta-derivative of f at t, denoted $f^\Delta(t)$, is defined to be the element of X with the property that given any $\epsilon > 0$, there is a neighborhood $\mathbf{U}$ of t such that

$$\left\|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right\| \leq \epsilon\left|\sigma(t) - s\right|, \quad \forall\, s \in \mathbf{U}.$$

If $f^\Delta(t)$ exists we say f is delta-differentiable at t, and we say $f^\Delta : \mathbb{T}^k \to X$ is the delta-derivative of f on $\mathbb{T}^k$. For the notion $\mathbb{T}^k$, see [3] page 2. We denote by

$$f^{\Delta\sigma} = (f^\Delta)^\sigma \text{ and } f^{\sigma\Delta} = (f^\sigma)^\Delta.$$

Throughout the rest of the article, we denote by

$$C^1_{rd}([a, b]_\mathbb{T}, X) = \{f : [a, b]_\mathbb{T} \to X | f^\Delta \text{ exists and rd-continuous }\},$$

and

$$C^2_{rd}([a, b]_\mathbb{T}, X) = \{f : [a, b]_\mathbb{T} \to X | f^\Delta, f^{\Delta^2} \text{ exist and rd-continuous }\}.$$

As usual for a bounded function f from $\mathbb{T}$ to X, we denote by

$$\|f\|_\infty = \sup_{t\in\mathbb{T}} \|f(t)\|.$$

2. MAIN RESULTS

In this section, assume that $p \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ and $f \in C_{rd}([a,b]_{\mathbb{T}}, X)$. We investigate Hyers-Ulam and Hyers-Ulam-Rassias stability of equation (1.1). First we recall the concept of Hyers-Ulam and Hyers-Ulam-Rassias stability. See [9].

Definition 2.1. (Hyers-Ulam stability)
We say that equation (1.1) has Hyers-Ulam stability if for any $\epsilon > 0$ and any $\psi \in C^2_{rd}([a,b]_{\mathbb{T}}, X)$ satisfies

$$\|\psi^{\Delta^2}(t) + p(t)\psi(t) - f(t)\| < \epsilon, \ \ t \in [a,b]_{\mathbb{T}},$$

there exists a solution $\phi \in C^2_{rd}([a,b]_{\mathbb{T}}, X)$ of equation (1.1) such that

$$\|\psi(t) - \phi(t)\| < L\epsilon, \ \ t \in [a,b]_{\mathbb{T}},$$

for some $L > 0$.

Definition 2.2. (Hyers-Ulam-Rassias stability)
Let $\mathcal{C}$ be a family of positive rd-continuous functions on $[a,b]_{\mathbb{T}}$. We say that equation (1.1) has Hyers-Ulam-Rassias stability of type $\mathcal{C}$, if for any $\omega \in \mathcal{C}$ and any $\psi \in C^2_{rd}([a,b]_{\mathbb{T}}, X)$ that satisfies

$$\|\psi^{\Delta^2}(t) + p(t)\psi(t) - f(t)\| < \omega(t), \ \ t \in [a,b]_{\mathbb{T}},$$

there exists a solution $\phi \in C^2_{rd}([a,b]_{\mathbb{T}}, X)$ of equation (1.1) such that

$$\|\psi(t) - \phi(t)\| < L\omega(t), \ \ t \in [a,b]_{\mathbb{T}},$$

for some $L > 0$.

We need the following lemma in proving our results.

Lemma 2.3. *ψ is a solution of equation (1.1) if and only if ψ satisfies the integral equation*

$$\psi(t) = a_0 + a_1(t-a) + \int_a^t (s - t + \mu(s))(p(s)\psi(s) - f(s))\Delta s. \qquad (2.1)$$

for some constants $a_0, a_1 \in X$.

Proof. Assume that ψ satisfies the integral equation (2.1). We denote by

$$M(t) = \int_a^t (s - t + \mu(s))(p(s)\psi(s) - f(s))\Delta s.$$

By Theorem 1.117 in [3], we conclude that

$$M^{\Delta}(t) = -\int_a^t (p(s)\psi(s) - f(s))\Delta s,$$

and

$$M^{\Delta^2}(t) = -(p(t)\psi(t) - f(t)).$$

This implies that $\psi^{\Delta^2}(t) = -(p(t)\psi(t) - f(t))$.

To prove the other direction, assume ψ is a solution of equation (1.1). We denote by

$$g(t) = p(t)\psi(t) - f(t),$$

$$G(t) = \int_a^t g(s)\,\Delta s,$$

and

$$L(t) = \int_a^t G(s)\,\Delta s.$$

Simple calculations show, by integrating two times both sides of (1.1), that

$$\psi(t) = a_0 + a_1(t-a) - L(t).$$

Here $a_i = \psi^{\Delta^i}(a), i = 0, 1$. It is readily seen that, $M(t) = -L(t)$ for every t. Indeed, we have

$$\begin{aligned} L^{\Delta}(t) &= G(t) \\ &= \int_a^t g(s)\Delta s \\ &= -M^{\Delta}(t). \end{aligned} \tag{2.2}$$

Consequently, $M(t) = -L(t) + C, t \in [a,\infty) \cap \mathbb{T}$. We have $C = M(a) + L(a) = 0$. Therefore ψ, satisfies equation (2.1). □

Corollary 2.4. *For any two elements* $a_0, a_1 \in X$*, equation* (1.1) *has at most one solution satisfying* $\psi^{\Delta^i}(a) = a_i, i = 0, 1$.

Proof. Assume that ψ_1 and ψ_2 are solutions of equation (1.1) which satisfy same initial conditions. Then both of them satisfy equation (2.1). This implies that

$$\|\psi_1(t) - \psi_2(t)\| \le \int_a^t |s - t + \mu(s)||p(s)|\|\psi_1(s) - \psi_2(s)\|\Delta s.$$

By Grönwal inequality [3], $\psi_1 = \psi_2$. □

Remark 2.5. It is well known that if equation (1.1) is regressive, that is $1+\mu^2(t)p(t) \neq 0$ for all $t \in [a,b]_{\mathbb{T}}$, then it has a unique solution x that satisfies the initial conditions $x^{\Delta^i}(a) = a_i, i = 0,1$ for every $a_0, a_1 \in X$. See [3].

Another sufficient condition for the existence of a unique solution of equation (1.1) can be stated in the following theorem.

Theorem 2.6. *If there is* $\alpha \in (0,1)$ *such that*

$$\int_a^t |p(s)|\,\Delta s \leq \frac{\alpha}{b-a+\|\mu\|_\infty}, t \in [a,b]_{\mathbb{T}},$$

then equation (1.1) *has a unique solution* ψ *that satisfies the initial conditions* $\psi^{\Delta^i}(a) = a_i, i = 0,1$ *for any* $a_0, a_1 \in X$.

Proof. Fix $a_0, a_1 \in X$. Define the operator $T : C_{rd}([a,b]_{\mathbb{T}}, X) \to C_{rd}([a,b]_{\mathbb{T}}, X)$ by

$$T\psi(t) = a_0 + a_1(t-a) + \int_a^t (s-t+\mu(s))(p(s)\psi(s) - f(s))\Delta s.$$

For $\phi, \psi \in C_{rd}([a,b]_{\mathbb{T}}, X)$, we have

$$\begin{aligned} \|T\psi(t) - T\phi(t)\| &\leq (b-a+\|\mu\|_\infty)\|\psi-\phi\|_\infty \int_a^t |p(s)|\,\Delta s \\ &\leq \alpha\|\psi-\phi\|_\infty, \ \ t \in [a,b]_{\mathbb{T}}. \end{aligned} \tag{2.3}$$

This implies that T is contraction. Therefore T has a unique fixed point ψ which is the solution of the integral equation (2.1) satisfying the initial conditions. □

The next theorem indicates the equivalence between the existence of a solution of the scalar homegeneous second order equation and the existence of a solution z of the corresponding Riccati equation. See also [3].

Theorem 2.7. *Assume that Riccati equation*

$$z^\Delta(1-\mu(t)z(t)) - z^2(t) = p(t), \ \ t \in [a,b]_{\mathbb{T}} \tag{2.4}$$

associated with the scalar equation

$$\psi^{\Delta^2}(t) + p(t)\psi(t) = 0, \ \ t \in [a,b]_{\mathbb{T}}, \tag{2.5}$$

has a solution z *that satisfies* $1-\mu(t)z(t) \neq 0, t \in [a,b]_{\mathbb{T}}$. *Then equation* (2.5) *has a solution. Conversily, if equation* (2.5) *has a solution with no zeros, then equation* (2.4) *has a solution.*

Proof. Let z be a solution of (2.4) that satisfies $1-\mu(t)z(t)\neq 0, t\in[a,b]_{\mathbb{T}}$. The function

$$x(t)=e_{-z}(t,a)$$

is a solution of the dynamic equation

$$x^{\Delta}(t)=-z(t)x(t).$$

In fact, for $t\in[a,b]_{\mathbb{T}}$, we have

$$\begin{aligned} x^{\Delta\Delta}(t)+p(t)x(t) &= z^2(t)x(t)-z^{\Delta}(t)x(\sigma(t))+p(t)x(t) \\ &= z^2(t)x(t)-z^{\Delta}(t)(x(t)+\mu(t)x^{\Delta}(t))+p(t)x(t) \\ &= z^2(t)x(t)-z^{\Delta}(t)(x(t)-\mu(t)z(t)x(t))+p(t)x(t) \\ &= x(t)\left(z^2(t)-z^{\Delta}(t)(1-\mu(t)z(t))+p(t)\right) \\ &= 0. \end{aligned} \tag{2.6}$$

Conversily, assume that x is a scalar solution of equation (2.5) with no zeros. Define z by

$$z(t)=-\frac{x^{\Delta}(t)}{x(t)}, t\in[a,b]_{\mathbb{T}}.$$

Then

$$z^{\Delta}(t)=-\frac{x(t)x^{\Delta\Delta}(t)-(x^{\Delta}(t))^2}{x(t)x^{\sigma}(t)}.$$

Simple calculations show that

$$\begin{aligned} z^{\Delta}(t)(1-\mu(t)z(t))-z^2(t) &= -\frac{x^{\Delta\Delta}(t)}{x(t)} \\ &= p(t), t\in[a,b]_{\mathbb{T}}. \end{aligned} \tag{2.7}$$

□

In the following result we establish a new sufficient condition for the existence of a unique solution x of the scalar equation (2.5) that satisfies the initial conditions $x^{\Delta^i}(a)=a_i, i=0,1$ for any $a_0,a_1\in\mathbb{R}$.

Theorem 2.8. *Assume that Riccati equation* (2.4) *has a solution* z *that satisfies* $1-\mu(t)z(t)\neq 0, t\in[a,b]_{\mathbb{T}}$. *Then equation* (2.5) *has a unique solution* x *that satisfies the initial conditions* $x^{\Delta^i}(a)=a_i, i=0,1$ *for any* $a_0,a_1\in\mathbb{R}$.

Proof. In view of Theorem 2.7, assume that $x(t)=e_{-z}(t,a)$ is a solution of equation (2.5). We investigate another solution y of the form

$$y(t)=u(t)x(t),$$

where u is a scalar function which will be chosen such that $\{x, y\}$ is a fundamental set for equation (2.5). We have

$$y^{\Delta}(t) = u(t)x^{\Delta}(t) + u^{\Delta}(t)x^{\sigma}(t),$$

and consequently

$$\begin{aligned} y^{\Delta^2}(t) + p(t)y(t) &= u(t)x^{\Delta^2}(t) + u^{\Delta}(t)x^{\Delta\sigma}(t) \\ &\quad + u^{\Delta}(t)x^{\sigma\Delta}(t) + u^{\Delta^2}(t)x^{\sigma^2}(t) + p(t)u(t)x(t) \\ &= u^{\Delta^2}(t)x^{\sigma^2}(t) + u^{\Delta}(t)x^{\Delta\sigma}(t) + u^{\Delta}(t)x^{\sigma\Delta}(t). \end{aligned} \tag{2.8}$$

Thus y is a solution of equation (2.5) if and only if u satisfies the following equation

$$u^{\Delta^2}(t)e_{-z}^{\sigma^2}(t,a) + u^{\Delta}(t)(e_{-z}^{\Delta\sigma}(t,a) + e_{-z}^{\sigma\Delta}(t,a)) = 0.$$

Setting $q(t) = \dfrac{e_{-z}^{\Delta\sigma}(t,a) + e_{-z}^{\sigma\Delta}(t,a)}{e_{-z}^{\sigma^2}(t,a)}$ and $v = u^{\Delta}$, the previous equation yields

$$v^{\Delta}(t) + q(t)v(t) = 0,$$

whose solution is given by

$$v(t) = e_{-q}(t,a).$$

Hence

$$u(t) = \int_a^t e_{-q}(s,a)\Delta s.$$

The Wronskian of x and $y = ux$ is given by

$$\begin{aligned} W(x,y)(t) &= e_{-z}(t,a)(u(t)e_{-z}^{\Delta}(t,a) + u^{\Delta}(t)e_{-z}^{\sigma}(t,a)) - e_{-z}(t,a)u(t)e_{-z}^{\Delta}(t,a) \\ &= e_{-z}(t,a)e_{-z}^{\sigma}(t,a)u^{\Delta}(t) \\ &= e_{-z}(t,a)e_{-z}^{\sigma}(t,a)e_{-q}(t,a). \end{aligned} \tag{2.9}$$

It follows that the Wronskian is positive and consequently $\{x, y\}$ is a fundamental set of equation (2.5). Then for any $a_0, a_1 \in \mathbb{R}$, there exist $c_1, c_2 \in \mathbb{R}$ such that the solution $\psi(t) = c_1x(t) + c_2y(t)$ of equation (2.5) satisfies the initial conditions $\psi^{\Delta^i}(a) = a_i, i = 0, 1.$ □

The following result establishes a new sufficient condition for the Hyers-Ulam stability of equation (1.1).

Theorem 2.9. *Assume that for any $a_0, a_1 \in X$ equation (1.1) has a unique solution ϕ that satisfies $\phi^{\Delta^i}(a) = a_i, i = 0, 1$. Then equation (1.1) has Hyers-Ulam stability.*

Proof. Let $\epsilon > 0$ and $\psi \in C^2_{rd}([a,b]_{\mathbb{T}}, X)$ satisfies

$$\|\psi^{\Delta^2}(t) + p(t)\psi(t) - f(t)\| < \epsilon, \quad t \in [a,b]_{\mathbb{T}}.$$

Set $h(t) = \psi^{\Delta^2}(t) + p(t)\psi(t) - f(t)$. Then ψ satisfies the equation

$$\psi^{\Delta^2}(t) + p(t)\psi(t) = f(t) + h(t), \quad t \in [a,b]_{\mathbb{T}}.$$

Let $a_i = \psi^{\Delta^i}(a), i = 0, 1$. Hence ψ satisfies

$$\psi(t) = a_0 + a_1(t-a) + \int_a^t [s - t + \mu(s)][p(s)\psi(s) - f(s) - h(s)]\Delta s.$$

There exists a unique solution ϕ of equation (1.1) satisfying $\phi^{\Delta^i}(a) = a_i, i = 0, 1$. Equivalently, ϕ satisfies the integral equation

$$\phi(t) = a_0 + a_1(t-a) + \int_a^t [s - t + \mu(s)][p(s)\phi(s) - f(s)]\Delta s. \tag{2.10}$$

We have

$$\begin{aligned}\|\psi(t) - \phi(t)\| &\leqslant \int_a^t |s - t + \mu(s)|\|h(s)\|\Delta s + \int_a^t |s - t + \mu(s)|\|p(s)(\psi(s) - \phi(s))\|\Delta s \\ &\leqslant (b-a)(b - a + \|\mu\|_\infty)\epsilon + (b - a + \|\mu\|_\infty)\|p\|_\infty \int_a^t \|\psi(s) - \phi(s)\|\Delta s.\end{aligned} \tag{2.11}$$

We denote by $K = (b - a + \|\mu\|_\infty)\|p\|_\infty$ and $M = \sup\{e_K(t,a) : t \in [a,b]_{\mathbb{T}}\}$. Inequality (2.11) yields

$$\|\psi(t) - \phi(t)\| \leq \frac{(b-a)K}{\|p\|_\infty}\epsilon + K\int_a^t \|\psi(s) - \phi(s)\|\Delta s. \tag{2.12}$$

By Grönwal's inequality [3], we deduce that

$$\|\psi(t) - \phi(t)\| \leq \frac{(b-a)KM}{\|p\|_\infty}\,\epsilon, \quad t \in [a,b]_{\mathbb{T}}. \tag{2.13}$$

Therefore, equation (1.1) is Hyers-Ulam stable. □

Since a regressive equation has a unique solution satisfying any initial conditions, [3], we get the following result

Theorem 2.10. *If equation* (1.1) *is regressive, then it has Hyers-Ulam stability.*

We combine theorems 2.6 and 2.9, to obtain a new sufficient condition for Hyers-Ulam stability of equation (1.1).

Theorem 2.11. *If there is $\alpha \in (0,1)$ such that*

$$\int_a^t |p(s)|\,\Delta s \le \frac{\alpha}{b-a+\|\mu\|_\infty}, \ \ t \in [a,b]_{\mathbb{T}},$$

then equation (1.1) has Hyers-Ulam stability.

We combine theorems 2.8 and 2.9 to obtain another sufficient condition for the Hyers-Ulam stability of the scalar equation (2.5).

Theorem 2.12. *The scalar equation (2.5) has Hyers-Ulam stability if the corresponding Ricatti equation (2.4) has a solution z that satisfies $1-\mu(t)z(t) \neq 0, t \in [a,b]_{\mathbb{T}}$.*

The following results are concerning with Hyers-Ulam-Rassias stability. Throughout the rest of the paper, we denote by

$$\mathcal{M} = \{\omega \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}) : \omega \text{ is positive, } \int_a^t \omega^2(s)\,\Delta s \le \omega^2(t)\}. \tag{2.14}$$

Theorem 2.13. *Assume that for any $a_0, a_1 \in X$ equation (1.1) has a unique solution ϕ that satisfies $\phi^{\Delta^i}(a) = a_i, i = 0,1$. Then equation (1.1) has Hyers-Ulam-Rassias stability of type $\mathcal{M}$.*

Proof. Let $\omega \in \mathcal{M}$ and $\psi \in C^2_{rd}([a,b]_{\mathbb{T}}, X)$ satisfies

$$\|\psi^{\Delta^2}(t) + p(t)\psi(t) - f(t)\| < \omega(t), \ \ t \in [a,b]_{\mathbb{T}}.$$

Set $h(t) = \psi^{\Delta^2}(t) + p(t)\psi(t) - f(t)$. Then ψ satisfies the equation

$$\psi^{\Delta^2}(t) + p(t)\psi(t) = f(t) + h(t), \ \ t \in [a,b]_{\mathbb{T}}.$$

Hence it satisfies the integral equation

$$\psi(t) = a_0 + a_1(t-a) + \int_a^t [s-t+\mu(s)][p(s)\psi(s) - f(s) - h(s)]\Delta s,$$

where $a_i = \psi^{\Delta^i}(a), i = 0,1$. There exists a unique solution ϕ of equation (1.1). Equivalently, ϕ satisfies the integral equation

$$\phi(t) = a_0 + a_1(t-a) + \int_a^t [s-t+\mu(s)][p(s)\phi(s) - f(s)]\Delta s. \tag{2.15}$$

In view of Hölder inequality [3], we have

$$\begin{aligned}\|\psi(t)-\phi(t)\| &\leqslant \int_a^t |s-t+\mu(s)|\|h(s)\|\Delta s + \int_a^t |s-t+\mu(s)||p(s)|\|\psi(s)-\phi(s)\|\Delta s \\ &\leqslant (b-a+\|\mu\|_\infty)\int_a^t \omega(s)\Delta s + (b-a+\|\mu\|_\infty)\|p\|_\infty \int_a^t \|\psi(s)-\phi(s)\|\Delta s \\ &\leqslant \sqrt{b-a}(b-a+\|\mu\|_\infty)\omega(t) + (b-a+\|\mu\|_\infty)\|p\|_\infty \int_a^t \|\psi(s)-\phi(s)\|\Delta s.\end{aligned} \tag{2.16}$$

We denote by

$$K = (b-a+\|\mu\|_\infty)\|p\|_\infty,$$

and

$$L = \sup_{t\in[a,b]_{\mathbb{T}}} \left(\int_a^b e_K^2(t,\sigma(s))\Delta s\right)^{\frac{1}{2}}$$

Inequality (2.16) yields

$$\|\psi(t)-\phi(t)\| \leq \frac{\sqrt{b-a}K}{\|p\|_\infty}\omega(t) + K\int_a^t \|\psi(s)-\phi(s)\|\Delta s. \tag{2.17}$$

By Grönwal's inequality [3] , we deduce that

$$\|\psi(t)-\phi(t)\| \leq \frac{\sqrt{b-a}K}{\|p\|_\infty}\,\omega(t) + \frac{\sqrt{b-a}K^2}{\|p\|_\infty}\int_a^t e_K(t,\sigma(s))\omega(s)\Delta s, \;\; t\in[a,b]_{\mathbb{T}}. \tag{2.18}$$

Again by Hölder inequality, it follows that

$$\|\psi(t)-\phi(t)\| \leq \frac{\sqrt{b-a}K}{\|p\|_\infty}\,\omega(t) + \frac{\sqrt{b-a}K^2L}{\|p\|_\infty}\left(\int_a^t \omega^2(s)\Delta s\right)^{\frac{1}{2}}, \;\; t\in[a,b]_{\mathbb{T}}. \tag{2.19}$$

This implies that

$$\|\psi(t)-\phi(t)\| \leq \frac{\sqrt{b-a}K}{\|p\|_\infty}\,(1+KL)\;\omega(t), \;\; t\in[a,b]_{\mathbb{T}}. \tag{2.20}$$

Therefore, eqation (1.1) is Hyers-Ulam-Rassias stable of type $\mathcal{M}$. □

Theorem 2.14. *If equation* (1.1) *is regressive, then it has Hyers-Ulam-Rassias stability of type* $\mathcal{M}$.

We combine theorem 2.6 and Theorem 2.13, to obtain a new sufficient condition for Hyers-Ulam-Rassias stability of equation (1.1).

Theorem 2.15. *If there is $\alpha \in (0,1)$ such that*

$$\int_a^t |p(s)|\,\Delta s \leq \frac{\alpha}{b-a+\|\mu\|_\infty}, \ \ t \in [a,b]_{\mathbb{T}},$$

then equation (1.1) *has Hyers-Ulam-Rassias stability of type $\mathcal{M}$.*

We combine theorems 2.8 and 2.13 to get another sufficient conditon for Hyers-Ulam-Rassias stability of the scalar equation (2.5).

Theorem 2.16. *The scalar equation* (2.5) *has Hyers-Ulam-Rassias stability of type $\mathcal{M}$ if the corresponding Ricatti equation* (2.4) *has a solution z that satisfies* $1-\mu(t)z(t) \neq 0, t \in [a,b]_{\mathbb{T}}$.

Remark 2.17. Theorems 2.13-2.16 hold if we replace $\mathcal{M}$ by

$$\mathcal{K} = \{\omega \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}) : \omega \text{ is non-negative, } \int_a^t \omega(s)\,\Delta s \leq \omega(t)\}. \tag{2.21}$$

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