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Existence Results for Semipositone Boundary Value Problems at Resonance

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Abstract

This work is concerned with the existence of three positive solutions for semipositone boundary value problems of three-point boundary conditions. The analysis is based upon a fixed point theorem on a cone.

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1 Introduction

Nonlinear boundary value problems (BVPs) for ordinary differential equations (ODEs) with nonlocal boundary conditions (BCs) have been well studied over the past decades. The study of nonlocal BCs for ODEs goes back, as far as we know, to Picone [13] and have been widely investigated during the years by many authors [2, 12, 15].

The research of nonlinear boundary value problems with multi-point boundary conditions at nonresonance case has a significant part in both theory and applications [4, 10, 16]. Moreover, it is known that the research of existence of solutions for nonlinear boundary value problems is not easy for the resonant case. Lately, the multi-point boundary value problems at resonance for ordinary differential equations have been investigated in detail and many successful results have been attained, for instance, see [1, 6, 7, 11]. Nevertheless, to our knowledge, the corresponding results for second-order with integral boundary conditions, are seldom seen [3, 8, 9] and very little research

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has been done on semipositone problems for nonlinear integral boundary value problems at resonance [5].

In [9], Liu and Ouyang established solutions for the nonlinear boundary value problems with integral boundary conditions

$$\begin{cases} z''(t) + f(t, z(t)) = 0, \quad 0 < t < 1, \\ z(0) = 0, \quad z(1) = \alpha \displaystyle\int_0^\eta z(s)ds, \end{cases}$$

where $\eta \in (0, 1)$, $\frac{1}{2}\alpha\eta^2 = 1$ and $f \in \mathcal{C}([0, 1] \times \mathbb{R}, \mathbb{R})$. The authors obtained a sufficient condition for the existence results of solutions.

In [5], Henderson and Kosmatov consider the Neumann boundary value problem at resonance

$$\begin{cases} -z''(t) = f(t, z(t)), 0 < t < 1, \\ z'(0) = z'(1) = 0, \end{cases}$$

where $\beta : [0, 1] \to \mathbb{R}^+$, $\alpha \neq 0$ and $f(t, z) + \alpha^2 z + \beta(z) \geq 0$, for $t \in [0, 1]$ and $z \geq 0$. Here positive solutions are obtained by means of the Guo–Krasnosel'skii fixed point theorem.

Motivated by the papers mentioned above, we study the existence of positive solutions for the following boundary value problem with a sign-changing nonlinearity

$$\begin{cases} z''(t) = f(t, z(t)), \quad t \in (0, 1), \\ z'(0) = 0, \quad z(1) = \alpha \displaystyle\int_0^n z(s)ds, \end{cases} \tag{1.1}$$

where $\alpha > 1$, $n \in (0, 1)$, $\alpha n = 1$, the continuous function $f : [0, 1] \times [0, \infty) \to \mathbb{R}$ satisfies the inequality $f(t, z) \geq -\beta^2 z - M$ in $[0, 1] \times [0, \infty)$, for some constant $\beta \neq 0$ and a positive constant $M > 0$.

When a nontrivial solution exists for the homogeneous boundary value problem, the corresponding boundary value problem is called at resonance.

Consider

$$\begin{cases} z''(t) + \beta^2 z(t) = F(t, z(t)), \quad t \in (0, 1), \\ z'(0) = 0, \quad z(1) = \alpha \displaystyle\int_0^n z(s)ds, \end{cases} \tag{1.2}$$

where

$$F(t, z) = f(t, z) + \beta^2 z.$$

It is easy to see that the BVP (1.1) is equivalent to the BVP (1.2) and the BVP (1.2) is at nonresonance.

Throughout the paper we will assume that the following conditions hold:

(H1) $\beta \in \left(\alpha, \dfrac{\pi}{2}\right)$;

(H2) $f : [0,1] \times [0,\infty) \to \mathbb{R}$ is a continuous function;

(H3) there exists $M > 0$, satisfying

$$F(t,z) + M \geq 0, \quad (t,z) \in [0,1] \times [0,\infty).$$

2 Preliminaries

Lemma 2.1. *Assume* $\alpha n = 1$. *If* $h \in \mathcal{C}([0,1],\mathbb{R})$, *then the BVP*

$$\begin{cases} z''(t) + \beta^2 z(t) = h(t), & t \in (0,1), \\ z'(0) = 0, \ z(1) = \alpha \displaystyle\int_0^n z(s)ds \end{cases} \tag{2.1}$$

has a unique solution

$$z(t) = \int_0^1 G(t,s)h(s)ds.$$

Here

$$G(t,s) = \begin{cases} \dfrac{1}{\beta}\sin\beta(t-s), & 0 \leq s \leq t \leq 1, \\ \\ 0, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$+\frac{\cos\beta t}{k\beta^2}\begin{cases} \alpha\cos\beta(n-s) + \beta\sin\beta(1-s) - \alpha, & s \leq n, \\ \\ \beta\sin\beta(1-s), & n \leq s, \end{cases}$$

where $k = \dfrac{\alpha}{\beta}\sin\beta n - \cos\beta$.

Lemma 2.2. *Suppose that condition* (H1) *is satisfied. Then*
i) *For any* $t, s \in [0,1]$, $G(t,s) \geq 0$.
ii) *There exist* $d \in (0,1]$ *and* $\phi \in \mathcal{C}([0,1],[0,\infty))$ *satisfying*

$$d\phi(s) \leq G(t,s) \leq \phi(s), \ t,s \in [0,1].$$

Proof. By Lemma 2.1, we show the proof of (i) and (ii).

$i)$ Using $\cos\beta n > \cos\beta$ for $\beta \in \Big(\alpha, \dfrac{\pi}{2}\Big)$,

$$k > \frac{\alpha}{\beta}\sin\beta n - \cos\beta n$$

and $\tan\beta n > \beta n$ for $\beta n > 0$ with $\alpha n = 1$, we obtain $k > 0$. Moreover, it is enough to show the case when $s \in [0, n]$ and $s \leq t$. Let

$$\varphi(s) = \alpha\cos\beta(n-s) + \beta\sin\beta(1-s) - \alpha, \;\; s \in [0, n].$$

Since $\varphi''(t) \leq 0$, by standard calculus, the function φ is concave on $(0, n)$. Using the inequality $\cos\dfrac{\beta}{2} \geq \sin\dfrac{\beta}{2}$ for $\beta \in \Big(\alpha, \dfrac{\pi}{2}\Big)$, we can derive that

$$\begin{aligned}\varphi(0) &= \alpha\cos\beta n + \beta\sin\beta - \alpha \\ &= \beta\sin\beta - 2\alpha\sin^2\frac{\beta n}{2} \\ &\geq \beta\sin\beta - 2\alpha\sin^2\frac{\beta}{2} \\ &> \alpha\Big(\sin\beta - 2\sin^2\frac{\beta}{2}\Big) \\ &\geq 0.\end{aligned} \tag{2.2}$$

Because $\varphi(n) \geq 0$ holds, together with (2.2) and the concavity of the function φ, $\varphi(s) \geq 0$ for $s \in [0, n]$. Hence, $G(t, s) \geq 0$, $t, s \in [0, 1]$.

$ii)$ Let

$$\Phi(s) = 1 - s, \quad K(t, s) = \lambda\Phi(s) - G(t, s).$$

Upper bounds.

We will prove that for $s \geq t$ and $s \leq t$, $K(t, s) \geq 0$, $t, s \in [0, 1]$, when $\lambda > 0$ is sufficiently large.

Case 1. If $s \geq t$ and $s \in [0, n]$, then

$$\begin{aligned}K(t, s) &= \lambda(1-s) - \frac{\cos\beta t}{k\beta^2}\Big(\beta\sin\beta(1-s) - 2\alpha\sin^2\frac{\beta(n-s)}{2}\Big) \\ &\geq \lambda(1-n) - \frac{\sin\beta}{k\beta}.\end{aligned}$$

If $s \in [n, 1]$, then

$$\begin{aligned}K(t, s) &= \lambda(1-s) - \frac{\cos\beta t}{k\beta^2}\beta\sin\beta(1-s) \\ &\geq \lambda(1-s) - \frac{\sin\beta(1-s)}{k\beta}.\end{aligned}$$

Take $\lambda \geq \lambda_1 := \dfrac{1}{(1-n)k}$, then $K(t,s) \geq 0$ holds for $s \geq t$.

Case 2. If $s \leq t$ and $s \in [0,n]$, then

$$\begin{aligned} K(t,s) &\geq \lambda(1-s) - \frac{\sin\beta}{\beta} - \frac{\cos\beta t}{k\beta^2}\Big(\beta\sin\beta(1-s) - 2\alpha\sin^2\frac{\beta(n-s)}{2}\Big) \\ &\geq \lambda(1-s) - \frac{\sin\beta}{\beta} - \frac{\cos\beta t}{k\beta^2}\beta\sin\beta(1-s) \\ &\geq \lambda(1-n) - \frac{2\sin\beta}{k\beta}, \end{aligned}$$

so, for $\lambda \geq \lambda_2 := \dfrac{2\sin\beta}{(1-n)k\beta}$, we have $K(t,s) \geq 0$ for $s \leq t$. If $s \in [n,1]$, then

$$\begin{aligned} K(t,s) &\geq \lambda(1-s) - \frac{1}{\beta}\sin\beta(1-s) - \frac{\cos\beta n}{k\beta}\sin\beta(1-s) \\ &\geq \lambda(1-s) - (1-s) - \frac{\cos\beta n}{k}(1-s) \\ &= (1-s)\Big(\lambda - 1 - \frac{\cos\beta n}{k}\Big), \end{aligned}$$

so, for $\lambda \geq \lambda_3 := 1 + \dfrac{\cos\beta n}{k}$, we have $K(t,s) \geq 0$ for $s \leq t$.

Let $\lambda^* \geq \max\{\lambda_1, \lambda_2, \lambda_3\}$, then $K(t,s) \geq 0$ for $(t,s) \in [0,1]\times[0,1]$ and we obtain $\lambda^*\Phi(s) \geq G(t,s)$. If we choose $\phi(s) := \lambda^*\Phi(s)$, then $G(t,s) \leq \phi(s),\ t,s \in [0,1]$.

Lower bounds.

We will prove that for $s \geq t$ and $s \leq t$, $K(t,s) \geq 0$, $t,s \in [0,1]$, when $\lambda > 0$ is sufficiently small.

Case 1. Let $\psi(s) = \alpha\cos\beta(n-s) + \beta\sin\beta(1-s) - \alpha$. Note that ψ is concave and $\psi(s) \geq \min\{\psi(0), \psi(n)\}$ for $s \in [0,n]$. If $s \geq t$ and $s \in [0,n]$, then

$$\begin{aligned} K(t,s) &= \lambda(1-s) - \frac{\cos\beta t}{k\beta^2}(\alpha\cos\beta(n-s) + \beta\sin\beta(1-s) - \alpha) \\ &\leq \lambda(1-s) - \frac{\cos\beta}{k\beta^2}\min\{\alpha\cos\beta n + \beta\sin\beta - \alpha,\ \beta\sin\beta(1-n)\} \\ &\leq \lambda - \frac{\cos\beta}{k\beta^2}\min\{\alpha\cos\beta n + \beta\sin\beta - \alpha,\ \beta\sin\beta(1-n)\}, \end{aligned}$$

so, for $\lambda \leq \lambda_4 := \dfrac{\cos\beta}{k\beta^2}\min\{\alpha\cos\beta n + \beta\sin\beta - \alpha,\ \beta\sin\beta(1-n)\}$, we have $K(t,s) \leq 0$ for $s \geq t$. If $s \in [n,1]$, then we obtain

$$\begin{aligned} K(t,s) &= \lambda(1-s) - \frac{\cos\beta t}{k\beta^2}\beta\sin\beta(1-s) \\ &\leq \lambda(1-s) - \frac{\cos\beta}{k\beta}\sin\beta(1-s). \end{aligned}$$

Since

$$h(x) = \begin{cases} \dfrac{\sin x}{x}, & 0 < x \leq \dfrac{\pi}{2}, \\ 1, & x = 0 \end{cases}$$

is continuous on $[0, \dfrac{\pi}{2}]$, we can get

$$k_0 := \min_{x \in [0, \frac{\pi}{2}]} h(x) > 0,$$

$$\begin{aligned} K(t,s) &\leq \lambda(1-s) - \frac{\cos\beta}{k\beta} k_0 \beta(1-s) \\ &= (1-s)\Big(\lambda - \frac{k_0 \cos\beta}{k}\Big), \end{aligned}$$

so, for $\lambda \leq \lambda_5 := \dfrac{k_0 \cos\beta}{k}$, we deduce that $K(t,s) \leq 0$ for $s \geq t$.

Case 2. For $s \leq t$ and $s \in [0, n]$,

$$\begin{aligned} K(t,s) &= \lambda(1-s) - \frac{\cos\beta t}{k\beta^2}(\alpha\cos\beta(n-s) + \beta\sin\beta(1-s) - \alpha) \\ &\leq \lambda(1-s) - \frac{\cos\beta}{k\beta^2}\min\{\alpha\cos\beta n + \beta\sin\beta - \alpha,\ \beta\sin\beta(1-n)\} \\ &\leq \lambda - \frac{\cos\beta}{k\beta^2}\min\{\alpha\cos\beta n + \beta\sin\beta - \alpha,\ \beta\sin\beta(1-n)\}, \end{aligned}$$

so, for $\lambda \leq \lambda_4 := \dfrac{\cos\beta}{k\beta^2}\min\{\alpha\cos\beta n + \beta\sin\beta - \alpha,\ \beta\sin\beta(1-n)\}$, we have $K(t,s) \leq 0$ for $s \leq t$. For $s \in [n, 1]$,

$$\begin{aligned} K(t,s) &\leq \lambda(1-s) - \frac{\cos\beta t}{k\beta}\sin\beta(1-s) \\ &\leq \lambda(1-s) - \frac{\cos\beta}{k\beta} k_0 \beta(1-s) \\ &= (1-s)\Big(\lambda - \frac{k_0\cos\beta}{k}\Big), \end{aligned}$$

so, for $\lambda \leq \lambda_5 := \dfrac{k_0\cos\beta}{k}$, we have $K(t,s) \leq 0$ for $s \leq t$. Choose $0 < \lambda_0 \leq \min\{\lambda_4, \lambda_5\}$. Then $K(t,s) \leq 0$ for $(t,s) \in [0,1] \times [0,1]$ and $\lambda_0 \Phi(s) \geq G(t,s)$. So,

$$d\phi(s) \leq G(t,s) \leq \phi(s),\ t, s \in [0,1],$$

where $d = \dfrac{\lambda_0}{\lambda^*}$. Therefore, the proof is completed. □

Theorem 2.3 (See [14]). *Assume that $\mathbb{B}$ is a Banach space, $K \subseteq \mathbb{B}$ is a cone in $\mathbb{B}$ and set*

$$\begin{aligned} K_r &= \{z \in K : \|z\| < r\}, \\ K(\varphi, a, b) &= \{z \in K : a \leq \varphi(z), ||z|| \leq b\}. \end{aligned}$$

Suppose that the operator $T : K \to K$ is completely continuous and φ is a nonnegative, continuous, concave functional on K with $\varphi(z) \leq ||z||$ for any $z \in \overline{K_r}$. If there exist $0 < p < q < d \leq c < r$ such that

(i) $\|Tz\| \leq c$ for $z \in \overline{K_c}$;

(ii) $\{z \in K(\varphi, q, d) : \varphi(z) > q\} \neq \emptyset$ *and* $\varphi(Tz) > q \ for \ all \ z \in K(\varphi, q, d)$;

(iii) $||Tz|| < p \ for \ all \ ||z|| \leq p$;

(iv) $\varphi(Tz) > q \ for \ z \in K(\varphi, q, c) \ with \ ||Tz|| > d$;

(v) $\|Tz\| \geq \|z\|$ for $z \in \partial K_r$.

Then T has at least four fixed points $z_1, z_2,\ z_3$ and z_4 in $\overline{K_r}$ such that

$$||z_1|| < p,\ \varphi(z_2) > q,\ \ p < ||z_3|| \ with \ \varphi(z_3) < q, c < \|z_4\| \leq r.$$

3 Main Result

In this section, by using Theorem 2.3, we obtain the existence of positive solutions for the BVP (1.1).

Here, we study on $\mathcal{C}[0, 1]$ with the norm $\|z\| = \sup_{t \in [0,1]} |z(t)|$, which is a Banach space.

Choose the cone by $P = \left\{ z \in \mathcal{C}[0, 1] : z(t) \geq 0, \min_{0 \leq t \leq 1} z(t) \geq d||z|| \right\}$. Apparently, P is a cone in $\mathcal{C}[0, 1]$.

Let $\varphi : P \to [0, \infty)$ be a nonnegative continuous concave functional defined by

$$\varphi(z) = \min_{t \in [0,1]} z(t), \quad \forall z \in P.$$

It is obvious that $\varphi(z) \leq ||z||, \forall z \in P$.

We need the following lemma.

Lemma 3.1. *Let v be a unique positive solution for the following boundary value problem*

$$\begin{cases} z''(t) + \beta^2 z(t) = 1, \quad t \in (0, 1), \\ z'(0) = 0, \ \ z(1) = \displaystyle\int_0^n z(s)ds. \end{cases} \tag{3.1}$$

Then

$$v(t) \leq dL, \quad t \in [0,1],$$

in which $L = \dfrac{K}{d}$, $K = \displaystyle\int_0^1 \phi(s)ds$.

Proof. Using the Green' s function, we get

$$\begin{aligned} v(t) &= \int_0^1 G(t,s)ds \\ &\leq \int_0^1 \phi(s)ds \leq dL. \end{aligned}$$

The proof is complete. □

For the readers convenience, let us set

$$\Delta = \min\left\{\frac{1}{ML}, 1\right\}.$$

Our main result is as follows:

Theorem 3.2. *Suppose that* (H1)–(H3) *are satisfied. Moreover there exist nonnegative constants* a, b, c, N *with* $1 < a < a+d < b < d^2c < c$, $\dfrac{1}{\lambda d} < N < \dfrac{c}{b}$ *satisfying*

(i) $f(t,z) + \beta^2 z + M < \dfrac{a}{K}$ *for* $t \in [0,1]$ *and* $z \in [0,a]$*;*

(ii) $f(t,z) + \beta^2 z + M \geq \dfrac{b}{K}N$ *for* $t \in [0,1]$ *and* $z \in \left[b-d, \dfrac{b}{d^2}\right]$*;*

(iii) $f(t,z) + \beta^2 z + M \leq \dfrac{c}{K}$ *for* $t \in [0,1]$ *and* $z \in [0,c]$*;*

(iv) $\liminf\limits_{z\to\infty} \min\limits_{t\in[0,1]} \dfrac{f(t,z)}{z} = \infty$.

Then the BVP (1.1) *has at least three positive solutions if* $\lambda \in (0,\Delta]$.

Proof. Assume that $x(t) = \lambda M v(t)$, in which v is a unique solution of the BVP (3.1). Lemma 3.1 implies that $x(t) = \lambda M v(t) \leq \lambda M L d \leq d$ for $t \in [0,1]$. Now, we prove that the following boundary value problem

$$\begin{cases} z''(t) + \beta^2 z(t) = \widetilde{F}(t, z(t) - x(t)), \quad t \in (0,1), \\ z'(0) = 0, \ \ z(1) = \displaystyle\int_0^n z(s)ds \end{cases} \tag{3.2}$$

has a positive solution, where

$$\widetilde{F}(t,u)=\begin{cases}F(t,u)+M, & u\geq 0\\ F(t,0)+M, & u\leq 0.\end{cases}$$

Denote the operator $A:E\to E$ by

$$Az(t)=\lambda\int_0^1 G(t,s)\widetilde{F}(s,z(s)-x(s))ds.$$

We shall prove that A has a fixed point in our cone P. Notice that $Az(t)$ is continuous on $[0,1]$, for any $z\in P$, and since $G(t,s)\geq 0,\ \ \phi(s)\geq 0$ holds. Then, by Lemma 2.2 we have

$$||Az|| \ \leq\ \lambda\int_0^1 \phi(s)\widetilde{F}(s,z(s)-x(s))ds. \tag{3.3}$$

Thus for any $z\in P$, we can deduce from (3.3) and Lemma 2.2 that

$$\begin{aligned}\min_{0\leq t\leq 1} Az(t) &= \min_{0\leq t\leq 1}\Big\{\int_0^1 \lambda G(t,s)\widetilde{F}(s,z(s)-x(s))ds\Big\}\\ &\geq d\int_0^1 \lambda\phi(s)\widetilde{F}(s,z(s)-x(s))ds\\ &\geq d||Az||.\end{aligned}$$

So, $A(P)\subset P$. Then from the definition of A, one can see easily that the operator A is completely continuous, and each fixed point of A in P is a solution of BVP (3.2).

Firstly, we shall show that $A:\overline{P_c}\to\overline{P_c}$.

Let $z\in\overline{P_c}$, then $||z||\leq c$. If $\max\{z(t)-x(t),0\}=z^*(t)$ then we get $\widetilde{F}(t,z^*(t))=f(t,z^*(t))+\beta^2 z^*(t)+M\geq 0$. We can derive from (iii) that $\widetilde{F}(t,z^*(t))\leq\dfrac{c}{K}$ for $t\in[0,1]$. So, it follows from (iii) that

$$\begin{aligned}Az(t)&=\int_0^1 \lambda G(t,s)\widetilde{F}(s,z(s)-x(s))ds\\ &\leq\lambda\int_0^1 \phi(s)\widetilde{F}(s,z(s)-x(s))ds\\ &\leq\lambda\frac{c}{K}\int_0^1\phi(s)ds\\ &\leq c.\end{aligned}$$

Then, $||Az||\leq c$ for $z\in\overline{P_c}$. So, condition (i) of Theorem 2.3 is satisfied. Similarly, using the above process, we can derive that $A:\overline{P_a}\to P_a$.

To verify (ii) of Theorem 2.3, let $z(t) = \frac{b}{d^2}$, then $z \in P\Big(\varphi, b, \frac{b}{d^2}\Big)$, $\varphi(z) = \frac{b}{d^2} > b$. Thus, $\Big\{z \in P\Big(\varphi, b, \frac{b}{d^2}\Big) : \varphi(z) > b\Big\} \neq \emptyset$. Furthermore, $z \in P\Big(\varphi, b, \frac{b}{d^2}\Big)$ implies that $b \leq z(t) \leq \frac{b}{d^2}$ for $t \in [0,1]$ and $b - d \leq z(t) - x(t) \leq z(t) \leq \frac{b}{d^2}$ for $t \in [0,1]$. (ii) implies that $F(t, z(t) - x(t)) + M \geq \frac{b}{K}N$ for $t \in [0,1]$. Then

$$\begin{aligned}\varphi(Az) = &\min_{t\in[0,1]} \lambda \int_0^1 G(t,s)\widetilde{F}(s, z(s) - x(s))ds \\ \geq &d\lambda \int_0^1 \phi(s)\widetilde{F}(s, z(s) - x(s))ds \\ = &d\lambda \int_0^1 \phi(s)(F(s, z(s) - x(s)) + M)ds \\ \geq &d\lambda \frac{b}{K} N \int_0^1 \phi(s)ds = d\lambda N b > b,\end{aligned}$$

i.e.,

$$\varphi(Az) > b, \quad \forall z \in P\Big(\varphi, b, \frac{b}{d^2}\Big).$$

On the other hand, if $z \in P(\varphi, b, c)$ with $||Az|| \geq \frac{b}{d^2}$, then we get

$$\varphi(Az) \quad = \quad \min_{t\in[0,1]} Az(t) \geq d||Az|| \geq \frac{b}{d} > b.$$

Finally, we will show that condition (v) of Theorem 2.3 holds. Choose a number $\gamma > 0$ satisfying $\frac{(\gamma + \beta^2)\lambda d^2 K}{2} \geq 1$. The hypotheses (iv) implies that there is $\rho > 0$ such that $1 - \frac{\lambda M L}{\rho} \geq \frac{1}{2}$ and

$$f(s, u(s)) + M \geq \gamma u(s), \; u(s) \geq \frac{d\rho}{2}. \tag{3.4}$$

Set $R = \max\left\{\frac{b}{d^2}, \rho\right\}$ and $P_R = \{z \in K : \|z\| < R\}$. Let $z \in \partial P_R$. Then

$$x(t) = \lambda M v(t) \leq \lambda M d L \leq \lambda M L \frac{z(t)}{R} \leq \frac{\lambda M L}{\rho} z(t), \; t \in [0,1].$$

Because

$$z(t) - x(t) \geq \frac{\rho - \lambda M L}{\rho} z(t) \geq \frac{z(t)}{2} \geq \frac{d\|z\|}{2} = \frac{dR}{2} \geq \frac{d\rho}{2}, \; t \in [0,1]$$

together with (3.4), we have

$$\begin{aligned}\widetilde{F}(s,z(s)-x(s)) &= f(s,z(s)-x(s))+\beta^2(z(s)-x(s))+M\\ &\geq (\gamma+\beta^2)(z(s)-x(s))\\ &\geq (\gamma+\beta^2)\frac{dR}{2}.\end{aligned}$$

Thus, we can obtain

$$\begin{aligned}Az(t) &\geq d\lambda\int_0^1 \phi(s)\widetilde{F}(s,z(s)-x(s))ds\\ &\geq \frac{(\gamma+\beta^2)\lambda d^2\int_0^1\phi(s)ds}{2}R\\ &\geq R.\end{aligned}$$

Then, $||Az|| \geq ||z||$ for $z \in \partial P_R$. Since all conditions of Theorem 2.3 are satisfied, $\widetilde{F}$ has at least four solutions $\overline{z_1}, \overline{z_2}, \overline{z_3}$ and $\overline{z_4}$ with $||\overline{z_1}|| < a$, $\varphi(\overline{z_2}) > b$, $||\overline{z_3}|| > a$, $\varphi(\overline{z_2}) < b$, $c < \|\overline{z_4}|| \leq R$. Besides,

$$\overline{z_2}(t) \geq d||\overline{z_2}|| \geq d\varphi(\overline{z_2}) > db > d > x(t), \quad t \in [0,1],$$

$$\overline{z_3}(t) \geq d||\overline{z_3}|| \geq da > d > x(t), \quad t \in [0,1],$$

and

$$\overline{z_4}(t) \geq d||\overline{z_4}|| \geq dc > d > x(t), \quad t \in [0,1].$$

Hence $z_2^* = \overline{z}_2 - x$, $z_3^* = \overline{z}_3 - x$, $z_4^* = \overline{z}_4 - x$ are positive solutions of the BVP (1.1).

Now we prove that z_2^*, z_3^*, and z_4^* are in fact the positive solutions of our problem (1.1). To see this we have for $t \in [0,1]$, $\overline{z_2}$ represents a fixed point of the operator $\widetilde{F}$. Then

$$\begin{aligned}\overline{z_2}(t) = \widetilde{F}\overline{z_2}(t) &= \int_0^1 \lambda G(t,s)\widetilde{F}(s,\overline{z_2}(s)-x(s))ds\\ &= \int_0^1 \lambda G(t,s)[F(s,\overline{z_2}(s)-x(s))+M]ds\\ &= \int_0^1 \lambda G(t,s)F(s,\overline{z_2}(s)-x(s))ds+\lambda M\int_0^1 G(t,s)ds\\ &= \int_0^1 \lambda G(t,s)F(s,\overline{z_2}(s)-x(s))ds+\lambda Mv(t)\\ &= \int_0^1 \lambda G(t,s)F(s,\overline{z_2}(s)-x(s))ds+x(t).\end{aligned}$$

This implies that

$$\overline{z_2}(t) - x(t) = \int_0^1 \lambda G(t,s) F(s, \overline{z_2}(s) - x(s)) ds$$

and

$$z_2^* = \int_0^1 \lambda G(t,s) F(s, z_2^*) ds.$$

Consequently, z_2^* is a positive solution of BVP (1.1). Similarly, we can show that z_3^* and z_4^* are positive solutions of the BVP (1.1). The proof is completed. □

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