Advances in Dynamical Systems and Applications
ISSN 0973-6069, Volume 13, Number 1, pp. 19–31 (2018)
http://campus.mst.edu/ijde

Boundedness and Stability of Solutions of Nonlinear Volterra Integro-Differential Equations

Fatimah Alahmadi, **Youssef N. Raffoul** and **Shuruq Alharbi**
University of Dayton
Department of Mathematics
Dayton, OH 45469-2316, USA

Abstract

We use Lyapunov functionals combined with Laplace transform and obtain boundedness and stability results regarding the solutions of the nonlinear Volterra integro-differential equation

$$y'(t) = A(t)y + f(y) + \int_0^t C(t,s)h(y(s))ds + p(t).$$

AMS Subject Classifications: 34D20, 39A10, 39A12, 40A05, 45J05.
Keywords: Volterra differential equation, nonlinear, boundedness, uniform, stability, Laplace transform, Lyapunov functionals.

1 Introduction

In this paper we are interested in the qualitative analysis of solutions for the nonlinear Volterra integro-differential equation

$$y'(t) = A(t)y + f(y) + \int_0^t C(t,s)h(y(s))ds + p(t),\ y(0) = y_0, \tag{1.1}$$

where A, $f(y)$, p, and $h(y)$ are scalar functions that are continuous. We assume the solution $y(t)$ of (1.1) to be continuous and never zero. It is clear that $y = 0$ is not a solution. In addition $C(t,s)$ is a scalar function on $\mathbb{R}^+ \times \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of all nonnegative real numbers. We are mainly interested in the boundedness of the solutions of (1.1) and the stability of its zero solution when $p(t) = 0$ for all $t \geq 0$.

Received August 25, 2017; Accepted January 15, 2018
Communicated by Martin Bohner

Throughout the paper we make the assumptions that for positive constants λ_1, λ_2 and M that

$$|f(y)| \leq \lambda_1 |y| , \tag{1.2}$$

$$|h(y)| \leq \lambda_2 |y| , \tag{1.3}$$

and

$$|p(t)| \leq M \text{ for all } t \geq 0. \tag{1.4}$$

Recently, several authors have studied the behavior of solutions of variant forms of (1.1). Medina [15–17], Eloe, Islam and Raffoul [6], and Raffoul [18], obtained stability and boundedness results of the solutions of the homogeneous part of (1.1) by means of representing the solution in terms of the resolvent matrix. Eloe and Murakami [6] and Elaydi et al. [5], used the notion of total stability and established results on the asymptotic behavior of the zero solution of (1.1). Their work heavily depended on showing or assuming the summability of the resolvent matrix. However, a major limitation of this procedure is that the resolvent matrix is an abstract term. When $f(y) = y$ and $|h(y)| \leq \lambda_2(t)\,|y|$, it was shown in [6] and [12], that the zero solution of (1.1) is uniformly asymptotically stable provided that $\int_0^\infty \lambda_2(t) < \infty$. In this research, we do not assume that $\int_0^\infty \lambda_2(t) < \infty$. For more results on stability of the zero solution of Volterra integro differential equation we refer the reader to Crisci, Komanovskii and Vecchio [3], Elaydi [4] and Agarwal, Pang [1] and the references therein. This research is a continuation of the research initiated in [9, 10, 12, 18] and related to the work in [6, 7, 11]. For recent results on Volterra integro-differential equations, we refer the reader to [8, 20–22] and the references there in.

2 Main Results

In this paper we intend to use Lyapunov functional $V(t)$ coupled with Laplace transform and obtain boundedness results concerning (1.1) and the stability of its zero solution when $p(t) = 0$ for all $t \geq 0$. At the end of the paper, we furnish two examples as application to our obtained results. A function $x(t)$ is of exponential order for $t \geq 0$ if there are constants $m \geq 0$ and c such that

$$|x| \leq me^{ct} \text{ for all } t \geq 0.$$

Moreover, if $x(t)$ is a piecewise continuous function defined for $t \geq 0$ of exponential order, then we define the Laplace transform $L(x)(s)$ of $x(t)$ is defined by the integral

$$L(x)(s) = \int_0^{+\infty} e^{-st} x(t) dt,$$

where s is a real number. For our one main results in Theorem 2.4, we assume the existence of a scalar, continuous and differentiable function $\varphi(t)$ such that

$$\varphi(t) \geq 0 \text{ and } \varphi'(t) \leq 0 \text{ for all } t \geq 0. \tag{2.1}$$

and

$$\int_0^\infty \varphi(t) < \infty. \tag{2.2}$$

In Theorem 2.4, we use Lyapunov functional of convolution type that allows us to apply Laplace transform to evaluate integral equations that are in convolution form. We begin with the following lemma which is crucial for proving Theorem 2.4.

Lemma 2.1. *Assume* (2.1) *and* (2.2) *hold. Let $\beta(t)$ be a scalar function that is uniformly continuous on $[0, \infty)$. For positive constant λ_3, let the scalar continuous function $H(t)$ be given by*

$$H(t) := \beta(t) + \lambda_3 \int_0^t \varphi(t-s)\beta(s)ds, \tag{2.3}$$

and

$$H'(t) = -\alpha\beta(t), \quad \alpha > 0, \quad \beta(0) = 1. \tag{2.4}$$

Then

$$\beta(t) + \int_0^t \{\lambda_3\varphi(t-s) + \alpha\}\beta(s)ds = 1, \tag{2.5}$$

$$\beta(t) > 0 \textit{ on } [0, \infty), \tag{2.6}$$

$$\beta(t) \in L^1[0, \infty), \tag{2.7}$$

and

$$\beta(t) \to 0 \quad \textit{as} \quad t \to \infty. \tag{2.8}$$

Proof. An integration of (2.4) from 0 to t gives

$$H(t) = H(0) - \alpha \int_0^t \beta(s)ds. \tag{2.9}$$

From (2.3) and (2.4), we obtain $H(0) = \beta(0) = 1$. Comparing (2.3) and (2.9) we arrive at

$$\beta(t) + \lambda_3 \int_0^t \varphi(t-s)\beta(s)ds = H(0) - \alpha \int_0^t \beta(s)ds.$$

Using the fact that $H(0) = \beta(0) = 1$ the above equation gives,

$$\beta(t) + \int_0^t (\lambda_3\varphi(t-s) + \alpha)\beta(s)ds = 1,$$

which proves (2.5).
Next we prove (2.6). Differentiate (2.5) with respect to t and get

$$\beta'(t) + (\lambda_3\varphi(0) + \alpha)\beta(t) = -\int_0^t \lambda_3\varphi'(t-s)\beta(s)ds,$$

which has the solution

$$\beta(t) = \beta(0)e^{-(\lambda_3\varphi(0)+\alpha)t} - \lambda_3\int_0^t\int_0^u \varphi'(u-s)\beta(s)dse^{-(\lambda_3\varphi(0)+\alpha)(t-u)}du.$$

Suppose β takes on negative values and since $\beta(0) = 1$ and it is uniformly continuous, it must cross the t-axis at some time t. Let t^* be such time. That is $\beta(t) > 0,\ 0 \le t \le t^*$ and $\beta(t^*) = 0$. Then in the double integral term, we have $\beta(s) \ge 0$, since $0 \le s \le t^*$. Thus, by letting $t = t^*$ we have

$$\beta(t^*) = \beta(0)e^{(\lambda_3\varphi(0)+\alpha)t^*} - \lambda_3\int_0^{t^*}\int_0^u \varphi'(u-s)\beta(s)dse^{-(\lambda_3\varphi(0)+\alpha)(t^*-u)}du > 0$$

since $\varphi'(u-s) \le 0$ we get $\beta(t^*) > 0$, which is a contradiction. This proves that $\beta(t) > 0$ for all $t \ge 0$. Expression (2.9) implies that

$$\alpha\int_0^t \beta(s)ds = H(0) - H(t) \le H(0) = 1.$$

Since $\beta(t) \ge 0$ for all $t \ge 0$, we have that H is monotonically decreasing by condition (2.9). Therefore

$$\int_0^t \beta(s)ds \le \frac{1}{\alpha} \qquad \text{for every } t.$$

Now as $t \to \infty$,

$$\int_0^\infty \beta(s)ds \le \frac{1}{\alpha},$$

which proves (2.7). The proof of (2.8) follows from [2], since β is uniformly continuous and $\beta(t) \in L^1[0,\infty)$. □

We are ready to state and prove our first theorem.

Theorem 2.2. *Assume the hypothesis of Lemma 2.1 along with* (1.2)–(1.4). *Suppose*

$$\lambda_2|C(t,s)| + \lambda_3\varphi'(t-s) \le 0 \qquad \textit{for } 0 \le s \le t < \infty \textit{ for } t \in \mathbb{R}, \tag{2.10}$$

and

$$A(t) + \lambda_1 + \lambda_3\varphi(0) \le -\alpha, \tag{2.11}$$

where α is a positive constant. Then all solutions of (1.1) *are bounded by* $V(0) + \frac{M}{\alpha}$. *Moreover,* $\lim_{t\to\infty}|y(t)| = \frac{M}{\alpha}$.

Proof. Define the Lyapunov functional V by

$$V(t) := |y| + \lambda_3 \int_0^t \varphi(t-s)|y(s)|ds, \qquad t \geq 0. \tag{2.12}$$

Note that

$$\frac{d}{dt}(|y(t)|) = \frac{d}{dt}(y^2(t))^{1/2} = y\frac{y'(t)}{(y^2(t))^{1/2}} = y(t)\frac{y'(t)}{|y(t)|}.$$

By differentiating $V(t)$, we obtain

$$V'(t) = y\frac{y'}{|y|} + \lambda_3\varphi(0)|y| + \lambda_3 \int_0^t \varphi'(t-s)|y(s)|ds, \qquad t \geq 0.$$

Then, substituting (1.1) in $V'(t)$ we have

$$V'(t) = \frac{y}{|y|}\left\{A(t)y(t) + f(y) + \int_0^t C(t-s)h(y(s))ds + p(t)\right\}$$
$$+\lambda_3\varphi(0)|y| + \lambda_3 \int_0^t \varphi'(t-s)|y(s)|ds,$$

which simplifies to

$$V'(t) = A(t)|y| + \frac{y}{|y|}f(y) + \frac{y}{|y|}\int_0^t C(t,s)h(y(s))ds + \frac{y}{|y|}p(t)$$
$$+\lambda_3\varphi(0)|y| + \lambda_3 \int_0^t \varphi'(t-s)|y(s)|ds$$

$$\leq A(t)|y| + |f(y)| + \int_0^t |C(t,s)||h(y(s))|ds + |p|$$
$$+\lambda_3\varphi(0)|y| + \lambda_3 \int_0^t \varphi'(t-s)|y(s)|ds.$$

Using (1.2)–(1.4), we have

$$V'(t) \leq A(t)|y| + \lambda_1|y| + \lambda_2 \int_0^t |C(t,s)||y(s)|ds + M$$
$$+\lambda_3\varphi(0)|y| + \lambda_3 \int_0^t \varphi'(t-s)|y(s)|ds.$$

After some algebra, we arrive at the simplified expression,

$$V(t)' \leq \Big\{[A(t) + \lambda_1 + \lambda_3\varphi(0)]\Big\}|y| +$$

$$\left\{ \int_0^t [\lambda_2 |C(t,s)| + \lambda_3 \varphi'(t-s)] |y(s)| ds \right\} + M. \tag{2.13}$$

Using (2.10) and (2.11) we arrive at

$$V'(t) \le -\alpha |y| + M, \qquad M > 0 \tag{2.14}$$

Taking the Laplace transform in (2.5)

$$L(\beta) + L\left(\int_0^t \lambda_3 \varphi(t-s)\beta(s) ds \right) + L\left(\int_0^t \alpha \beta(s) ds \right) = L(1),$$

$$\Rightarrow \quad L(\beta) + \lambda_3 L(\varphi * \beta) + \alpha L(1 * \beta) = \frac{1}{s}.$$

$$\Rightarrow \quad L(\beta) + \lambda_3 L(\varphi) L(\beta) + \alpha L(1) L(\beta) = \frac{1}{s}.$$

Solving for $L(\beta)$ gives

$$L(\beta) = \frac{1}{(1 + \lambda_3 L(\varphi) + \alpha \frac{1}{s})s} \tag{2.15}$$

Due to (2.14) there is a nonnegative function $\eta : [0, \infty) \to [0, \infty)$ such that

$$V'(t) := -\alpha |y| + M - \eta(t).$$

Since η is a linear combination of functions of exponential order, η, is also of exponential order and so we can take the Laplace transform and have

$$sL(V) - V(0) = -\alpha L(|y|) + \frac{M}{s} - L(\eta),$$

We get

$$L(V) = \left[V(0) - \alpha L(|y|) + \frac{M}{s} - L(\eta) \right] \frac{1}{s}.$$

Using (2.12), we have

$$L(V) = L(|y|) + \lambda_3 L(\varphi) L(|y|).$$

Setting $L(V) = L(V)$ and solving for $L(|y|)$, we get

$$L(|y|) = \frac{V(0) + \frac{M}{s} - L(\eta)}{(1 + \lambda_3 L(\varphi) + \frac{\alpha}{s})s}$$

or

$$L(|y|) = \left[V(0) + \frac{M}{s} - L(\eta) \right] \frac{1}{(1 + \lambda_3 L(\varphi) + \frac{\alpha}{s})s}.$$

Using (2.15)

$$L(|y|) = \left[V(0) + \frac{M}{s} - L(\eta) \right] L(\beta).$$

Then, we obtain

$$L(|y|) = L(\beta)V(0) + L(\beta)\frac{M}{s} - L(\beta)L(\eta). \tag{2.16}$$

Using the fact that

$$L(f * g) = L(f)L(g),$$

we obtain from (2.16) that

$$L(|y|) = L(\beta)V(0) + L(M * \beta) - L(\eta * \beta). \tag{2.17}$$

Since

$$\int_0^t f(s)g(t-s)ds = f * g$$

where $f(t)$ and $g(t)$ are piecewise continuous functions on $[0, \infty)$, we get

$$L(|y|) = L(\beta)V(0) + L\left(\int_0^t M\beta(s)ds\right) - L\left(\int_0^t \eta(t-s)\beta(s)ds\right). \tag{2.18}$$

Taking the inverse Laplace transform in (2.18), we get

$$|y| \le \beta(t)V(0) + M\int_0^t \beta(s)ds - \int_0^t \eta(t-s)\beta(s)ds$$

or

$$|y| \le \beta(t)V(0) + M\int_0^t \beta(s)ds. \tag{2.19}$$

By (2.5) we have $\beta(t) \le 1$, for all $t \ge 0$. This implies along with (2.19) that

$$\begin{aligned} |y(t)| &\le V(0) + M\int_0^\infty \beta(s)ds \\ &\le V(0) + \frac{M}{\alpha} \\ &= |y_0| + \frac{M}{\alpha}. \end{aligned}$$

Also, using (2.8) along with (2.19) we have that

$$\lim_{t\to\infty} |y(t)| = \frac{M}{\alpha}.$$

This completes the proof. □

Now we offer an example as an application.

Example 2.3. Let $p(t) = \cos(t)$ and

$$f(y) = \frac{1}{16}\sin(y), h(y) = \sin(y), \text{ and } C(t,s) = e^{-(t+4-s)}.$$

Then we have, $|f(y)| \leq \frac{1}{16}$, $|h(y)| \leq 1$, $|C(t,s)| = e^{-(t+4-s)}$, and $|p| \leq 1$.

Let $\phi(t) = e^{-(t+3)}$, $A(t) = -1$, and define the Lyapunov functional V by

$$V(t) = |y(t)| + \lambda_3 \int_0^t \varphi(t-s)|y(s)|ds.$$

Then we have $\lambda_1 = \frac{1}{16}, \quad \lambda_2 = 1, \quad$ and $\quad \lambda_3 = 1$. Since $\varphi'(t) = -e^{-(t+3)} \Rightarrow \phi'(t) \leq 0$, we get

$$\lambda_2|c(t,s)| + \lambda_3\varphi'(t-s) = e^{-(t+4-s)} - e^{-(t+3-s)} \leq 0 \text{ for } 0 \leq s \leq t < \infty.$$

Moreover, condition (2.11) is satisfied for $A(t)+\lambda_1+\lambda_2\varphi(0) \leq -\alpha$ where $\alpha = 0.8885$. Thus, by Theorem 2.4, all solutions of

$$y'(t) = -y(t) + \frac{1}{16}\sin(y(t)) + \int_0^t e^{-(t+4-s)}\sin(y(s))ds + \cos(t), \; y(0) = y_0,$$

are bounded and satisfy

$$|y| \leq |y_0| + \frac{1}{0.8885}, \text{ and } \lim_{t\to\infty} |y(t)| = \frac{1}{0.8885}.$$

In the next theorem we will study stability and boundedness of all solutions of (1.1) where $p(t) = 0$. Hence, we consider the nonlinear Volterra integro-differential equation

$$y'(t) = A(t)y + f(y) + \int_0^t C(t,s)h(y(s))ds, \; y(0) = y_0, \tag{2.20}$$

where, $f(0) = h(0) = 0$.

Theorem 2.4. *Assume* (1.2) *and* (1.3). *Also, Assume* $A(t) \leq 0$ *and*

$$|A(s)| - \lambda_1 - \lambda_2 \int_s^t |C(u,s)|du \geq 0 \quad \text{for } 0 \leq s \leq t \leq \infty. \tag{2.21}$$

Then the zero solution of (2.22) is stable. If in addition, there is a $t_2 \geq 0$ *and an* $\alpha > 0$ *with*

$$|A(s)| - \lambda_1 - \lambda_2 \int_s^t |C(u,s)|du \geq \alpha$$

for $t_2 \leq s \leq t \leq \infty$*, where* λ_1 *and* λ_2 *are positive, and if both*

$$|A(s)| \text{ and } \int_s^t |C(u,s)|du$$

are bounded, then the zero solution of (2.22) is asymptotically stable.

Proof. We begin by integrating (2.20) from 0 to t to get the Lyapunov functionals H. Thus,

$$\int_0^t y'(s)ds = \int_0^t A(s)y(s)ds + \int_0^t f(y(s))ds + \int_0^t \int_0^u C(u,s)h(y(s))dsdu.$$

Then, we obtain

$$y(t) = y(0) + \int_0^t A(s)y(s)ds + \int_0^t f(y(s))ds + \int_0^t \int_0^u C(u,s)h(y(s))dsdu. \quad (2.22)$$

Interchanging the order of the integration we arrive at

$$\int_0^t \int_0^u C(u,s)h(y(s))dsdu = \int_0^t \int_s^t C(u,s)h(y(s))duds. \quad (2.23)$$

Substituting (2.23) in (2.22) , we get

$$y(t) = y(0) + \int_0^t A(s)y(s)sd + \int_0^t f(y(s))ds + \int_0^t \int_s^t C(u,s)h(y(s))duds,$$

which is a solution of (2.20) on $[0,\infty)$. Define the Lyapunov functional $H(t, y(\cdot))$ by

$$H(t, y(\cdot)) = |y| + \int_0^t \left(|A(s)| - \lambda_1 - \lambda_2 \int_s^t |V(u,s)|du \right) |y(s)|ds. \quad (2.24)$$

It is clear that $H(t)$ given by (2.24) is positive since $|A(s)| - \lambda_1 - \lambda_2 \int_s^t |c(u,s)|du \geq 0$ for $0 \leq s \leq t < \infty$. Let $y(t) = y(t, t_0, y)$ be a solution of (2.20) on$[t_0, \infty]$, then by deriving $H(t, y(\cdot))$ we get

$$H'(t, y(\cdot)) = \frac{y}{|y|}y' + \left\{ |A| - \lambda_1 - \lambda_2 \int_t^t |C(u,t)|du \right\} |y(t)|$$

$$+ \int_0^t \frac{d}{dt} \left(\left(|A(s)| - \lambda_1 - \lambda_2 \int_s^t |C(u,s)|du \right) |y(s)| \right) ds.$$

Doing some algebra and using (2.20) , we get

$$\begin{aligned} H'(t, y(\cdot)) &= \frac{y}{|y|} \left(A(t)y(t) + f(y(t)) + \int_0^t C(t,s)(h(y(s))ds \right) \\ &+ |A||y| - \lambda_1|y| + \int_0^t (0 - \lambda_2 |C(t,s)||y(s)|) \, ds. \end{aligned}$$

Also, after some algebra

$$H'(t, y(\cdot)) = A(t)|y| + \frac{y}{|y|} f(y(t)) + \frac{y}{|y|} \int_0^t C(t,s)h(y(s))ds$$

$$+ \quad |A||y| - \lambda_1|y| - \lambda_2 \int_0^t |C(t,s)||y(s)|ds$$

$$\le A(t)|y| + \frac{|y|}{|y|}|f(y)| + \frac{|y|}{|y|}\int_0^t |C(t,s)||h(y(s))|ds$$

$$+|A||y| - \lambda_1|y(t)| - \lambda_2 \int_0^t |C(t,s)||(y(s))|ds$$

$$\le A(t)|y| + \lambda_1|y| + \lambda_2 \int_0^t |C(t,s)||y(s)|ds$$

$$+|A(t)||y| - \lambda_1|y| - \lambda_2 \int_0^t |C(t,s)||y(s)|ds = 0.$$

Hence, $H(t, y(\cdot))$ is decreasing. Now, we prove that the zero solution of (2.20) is stable. So for given $\epsilon > 0$ and $t_0 \ge 0$, let $\phi : [0, t_0] \to R$ be an initial continuous function with $|\phi| < \delta$ and $\delta > 0$ to be determined. Since

$$|y| \le H(t, y(\cdot))$$

and the fact that $H(t)$ is decreasing we have that

$$|y(t)| \le H(t, y(\cdot)) \le H(t_0, \phi(\cdot)).$$

This translates into

$$|y| \le |\phi| + \int_0^{t_0} \left[|A(s)| - \lambda_1 - \lambda_2 \int_s^{t_0} |C(u,s)|du\right] |\phi(s)|ds$$

$$< \delta + \delta \int_0^{t_0} \left[|A(s)| - \lambda_1 - \lambda_2 \int_s^{t_0} |C(u,s)|du\right] ds$$

$$= \delta \left\{ 1 + \int_0^{t_0} \left[|A(s)| - \lambda_1 - \lambda_2 \int_s^{t_0} |C(u,s)|du\right] ds \right\}$$

$$< \epsilon \quad \text{where } \delta = \frac{1}{\left\{ 1 + \int_0^{t_0} \left[|A(s)| - \lambda_1 - \lambda_2 \int_s^{t_0} |C(u,s)|du\right] ds \right\}}$$

if $\delta = \delta(\varepsilon, t_0)$is small enough. Therefore, the zero solution of (2.20) is stable. If $t_2 \ge 0$ and $\alpha > 0$ exists with

$$|A(s)| - \lambda_1 - \lambda_2 \int_s^{t_0} |C(u,s)|du \ge \alpha \text{ for } t_2 \le s \le t < \infty,$$

then

$$
\begin{aligned}
|y| + \int_{t_2}^{t} \alpha |y(s)| ds &\leq |y| + \int_{t_2}^{t} \left[|A(s)| - \lambda_1 - \lambda_2 \int_s^t |C(u,s)| du \right] |y(s)| ds \\
&\leq |y| + \int_{0}^{t} \left[|A(s)| - \lambda_1 - \lambda_2 \int_s^t |C(u,s)| du \right] |y(s)| ds \\
&= H(t, y(\cdot)) \leq H(t_0, \phi(\cdot)) < N \quad \text{where } N > 0.
\end{aligned}
$$

Therefore,

$$\int_{t_0}^{t} |y(s)| ds < \frac{N}{\alpha} = R \text{ where } R > 0,$$

and hence $y \in L^1[0, \infty)$. We also have $y'(t)$ bounded so $y(t) \to 0$ as $t \to \infty$. This completes the proof. □

Example 2.5. Let all the functions be given as in Example1.
As before we have that $\lambda_1 = \frac{1}{16}$, $\lambda_2 = 1$. Let $A(t) = -1$ and define the Lyapunov functional $H(t, y(\cdot))$ by

$$H(t, y(\cdot)) = |y(t)| + \int_0^t \left(|A(s)| - \lambda_1 - \lambda_2 \int_s^t |C(t,s)| du \right) |y(s)| ds.$$

Then,

$$
\begin{aligned}
|A(s)| - \lambda_1 - \lambda_2 \int_s^t |C(u,s)| du &= 1 - \frac{1}{16} - \int_s^t e^{-(u+4-s)} du \\
&= \frac{15}{16} + e^{-(t+4-s)} - e^{-4} \quad \text{where } 0 \leq s \leq t < \infty \\
&\geq \frac{15}{16} - e^{-4} = 0.9191 > 0.
\end{aligned}
$$

Hence, condition (2.21) is satisfied, so the zero solution of (2.20) is stable.

$$\int_s^t |C(u,s)| du = \int_s^t e^{-(u+4-s)} du = -e^{-(t+4-s)} + e^{-4} \leq e^{-4},$$

and so, $A(t)$ and $\int_s^t |C(u,s)| du$ are bounded. Hence, condition

$$|A(s)| - \lambda_1 - \lambda_2 \int_s^t |C(u,s)| du \geq \alpha \text{ for } 0 \leq t_2 \leq s \leq t < \infty$$

of Theorem 2.4 is satisfied for $\alpha = 0.9191$, so the zero solution of (2.20) is asymptotically stable.

Acknowledgement

The authors would like to thank the anonymous referee for his or her suggestions and careful reading of our paper.

References

[1] R. Agarwal, and P. Y. Pang, *On a generalized difference system*, Nonlinear Anal., TM and Appl., 30(1997), 365–376.

[2] Barbalat, *Systemes d'equations differentielles d'oscillations non Lineaires,* Rev. Math. Pures Appl. 4 (1959), 267–270.

[3] M. R. Crisci, V. B. Kolmanovskii, and A. Vecchio, *Boundedness of discrete Volterra equations*, J. Math. Analy. Appl. 211(1997), 106–130.

[4] S. Elaydi, *Periodicity and stability of linear Volterra difference systems*, J. Math. Analy. Appl. 181(1994), 483–492.

[5] S. Elaydi, and S. Murakami, *Uniform asymptotic stability in linear Volterra difference equations*, J. Difference Equ. Appl. 3(1998), 203–218.

[6] P. Eloe, M. Islam, and Y. Raffoul, *Uniform asymptotic stability in nonlinear Volterra discrete systems*, Comput. Math. Appl. Special Issue Advances in Difference Equations IV, 45 (2003), pp. 1033–1039.

[7] G. Eid, B.Ghalayani, and Y. Raffoul, *Lyapunov functional and stability in nonlinear finite delay Volterra discrete systems*, Int. J. Difference Equ, 2015, 10(1), 77–90.

[8] J. Graef, C. Tunc, and S. Evgin, *Behaviors of solutions of nonlinear functional Volterra integro-differential equations with multiple delays,* Dynam. Systems Appl. 25 (2016), no. 1-2, 39–46.

[9] M. Islam, and Y. Raffoul, *Uniform asymptotic stability in linear Volterra difference equations*, PanAmerican Mathematical Journal, 11(2001), No. 1, 61–73.

[10] M. Islam, and Y. Raffoul, *Stability properties of linear Volterra Discrete systems with nonlinear perturbation*, Communications in Applied Analysis, Vol. 7, No. 3 (2003), pp. 405–416.

[11] M. Islam, Y. Raffoul, *Bounded Solutions of Almost Linear Volterra Equations*, Adv. Dyn. Syst. Appl. Volume 7, Number 2(2012).

[12] T. Khandaker, and Y. Raffoul, *Stability properties of linear Volterra integrodifferential equations with nonlinear perturbation*, Difference Equ. Appl. (2002), Vol. 8, pp. 857–874.

[13] W. Kelley, and A. Peterson, *Difference equations an introduction with applications*, Second Edition, Academic Press, New York, 2001.

[14] L. Lakshmikantham, and D. Trigiante, *Theory of difference equations:Numerical methods and applications*, Academic Press, New York, 1991.

[15] R. Medina, *Asymptotic behavior of Volterra difference equations*, Comput. Math. Appl. Volume 41, Issues 5-6, March 2001, Pages 679–687.

[16] R. Medina, *The asymptotic behavior of the solutions of a Volterra difference equation*, Computers Math. Appl. 181(1994), 19–26.

[17] R. Medina, *Solvability of discrete Volterra equations in weighted spaces*, Dynamic Systems and Appl. 5(1996), 407–422.

[18] Y. Raffoul, *Boundedness and periodicity of Volterra systems of difference equations*, J. Difference Equ. Appl. 4(1998), 381–393.

[19] Y. Raffoul, *General theorems for stability and Boundedness for nonlinear functional discrete systems*, J. Math. Analy. Appl. 279 (2003), pp. 639–650.

[20] C. Tunc, *New stability and boundedness results to Volterra integro-differential equations with delay,* J. Egyptian Math. Soc. 24 (2016), no. 2, 210–213.

[21] C. Tunc, *A note on the qualitative behaviors of non-linear Volterra integro-differential equation,* J. Egyptian Math. Soc. 24 (2016), no. 2, 187–192.

[22] C. Tunc, *Properties of solutions to Volterra integro-differential equations with delay,* Appl. Math. Inf. Sci. 10 (2016), no. 5, 1775–1780.