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Weak Solutions for Impulsive Implicit Hadamard Fractional Differential Equations

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Abstract

In this article, we present some results concerning the existence of weak solutions for a class of functional impulsive implicit differential equations involving the Hadamard fractional order derivative in Banach spaces. The main results are proved by applying fixed point theory combined with the technique of measure of weak noncompactness.

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1 Introduction

Fractional differential and integral equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering and other applied sciences [28,37]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs of Abbas et al. [5,6], Kilbas et al. [30] and Zhou [40].

The measure of weak noncompactness is introduced by De Blasi [24]. The strong measure of noncompactness was developed first by Banaš and Goebel [14] and subsequently developed and used in many papers; see for example, Akhmerov et al. [10], Alvàrez [12], Benchohra et al. [21], Guo et al. [26], and the references therein. In [21,34] the authors considered some existence results by applying the techniques of the measure of noncompactness. Recently, several researchers obtained other results by application of the technique of measure of weak noncompactness; see [6, 18, 19], and the references therein.

Impulsive differential equations have become more important in recent years in some mathematical models of real phenomena, especially in biological or medical domains, in control theory, see for example the monographs of Abbas et al. [5], Benchohra et al. [20], Bainov and Simeonov [13], Graef et al. [25], Perestyuk et al. [35], and several papers have been published, see the papers of Abbas et al. [2–4], Agarwal et al. [8], Benchohra and Berhoun [16], and the references therein.

Implicit functional differential equations have been considered by many authors [7, 15,32,39]. Recently, considerable attention has been given to the existence of solutions of fractional differential equations with Hadamard fractional derivative; see [1,9,11,17, 22,38].

In this paper, our intention is to extend the results to implicit impulsive differential equations of Hadamard fractional derivative. We discuss the existence of weak solutions for the implicit impulsive Hadamard fractional differential equation of the form

$$\begin{cases} ({}^H D^r_{t_k} u)(t) = f(t, u(t), ({}^H D^r_{t_k} u)(t));\ t \in J_k,\ k = 0, \dots, m, \\ \\ \dfrac{(\ln t)^{r-1}}{\Gamma(r)} ({}^H I^{1-r}_{t_k} u)(t_k^+) = u(t_k^-) + L_k(u(t_k^-));\ k = 1, \dots, m, \\ \\ ({}^H I^{1-r}_1 u)(t)|_{t=1} = \phi, \end{cases} \tag{1.1}$$

where $T > 1,\ \phi \in E,\ J_0 = [1, t_1],\ J_k := (t_k, t_{k+1}];\ k = 1, \dots, m,\ 1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T,\ f : J_k \times E \times E \to E;\ ;\ k = 1, \dots, m,\ L_k : E \to E;\ k = 1, \dots, m$ are given continuous functions, E is a real (or complex) Banach space with norm $\|\cdot\|_E$ and dual E^*, such that E is the dual of a weakly compactly generated Banach space X, $\ln = \log_e$, ${}^H I^r_{t_k}$ is the left-sided mixed Hadamard integral of order $r \in (0, 1]$, and ${}^H D^r_{t_k}$ is the Hadamard fractional derivative of order r.

Our goal in this paper is to give existence results for implicit impulsive Hadamard fractional differential equations.

2 Preliminaries

Let $\mathcal{C}$ be the Banach space of all continuous functions v from $J := [1, T]$ into E with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t\in J} \|v(t)\|_E.$$

As usual, $AC(J)$ denotes the space of absolutely continuous functions from J into E. By $C_{r,\ln}(J)$, we denote the weighted space of continuous functions defined by

$$C_{r,\ln}(J) = \{w(t) : (\ln t)^r w(t) \in \mathcal{C},\ \|w\|_{C_{r,\ln}} := \sup_{t\in J} \|(\ln t)^r w(t)\|_E\}.$$

Let $(E, w) = (E, \sigma(E, E^*))$ be the Banach space E with its weak topology. Consider the Banach space

$$PC = \big\{u : J \to E : u \in C(J_k);\ k = 0, \ldots, m, \text{ and there exist } u(t_k^-) \\ \text{and } ({}^H I_{t_k}^{1-r} u)(t_k^+);\ k = 1, \ldots, m, \text{ with } u(t_k^-) = u(t_k)\big\},$$

with the norm

$$\|u\|_C = \sup_{t\in J} \|u(t)\|_E.$$

Also, we can define the weighted space of PC by

$$PC_{r,\ln}(I) = \{w(t) : (\ln t)^r w(t) \in PC,\ \|w\|_{PC_{r,\ln}} := \sup_{t\in J} \|(\ln t)^r w(t)\|_E\}.$$

In the following we denote $\|w\|_{PC_{r,\ln}}$ by $\|w\|_{PC}$.

Definition 2.1. A Banach space X is called weakly compactly generated (WCG, in short) if it contains a weakly compact set whose linear span is dense in X.

Definition 2.2. A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any (u_n) in E with $u_n \to u$ in (E, w) then $h(u_n) \to h(u)$ in (E, w)).

Definition 2.3 (See [36])**.** The function $u : I \to E$ is said to be Pettis integrable on J if and only if there is an element $u_j \in E$ corresponding to each $j \subset J$ such that $\phi(u_j) = \int_j \phi(u(s))ds$ for all $\phi \in E^*$, where the integral on the right hand side is assumed to exist in the sense of Lebesgue, (by definition, $u_j = \int_j u(s)ds$).

Let $P(J,E)$ be the space of all E–valued Pettis integrable functions on J, and $L^1(J,E)$, be the Banach space of measurable functions $u : J \to E$ which are Bochner integrable. Define the class $P_1(J,E)$ by

$$P_1(J,E) = \{u \in P(J,E) : \varphi(u) \in L^1(J,\mathbb{R}); \text{ for every } \varphi \in E^*\}.$$

The space $P_1(J,E)$ is normed by

$$\|u\|_{P_1} = \sup_{\varphi \in E^*,\ \|\varphi\| \le 1} \int_1^T |\varphi(u(x))| d\lambda x,$$

where λ stands for a Lebesgue measure on J.

The following result is due to Pettis (see [36, Theorem 3.4 and Corollary 3.41]).

Proposition 2.4 (See [36])**.** *If $u \in P_1(J,E)$ and h is a measurable and essentially bounded E–valued function, then $uh \in P_1(J,E)$.*

For all what follows, the sign " $\int$ " denotes the Pettis integral. Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [27,30] for a more detailed analysis.

Definition 2.5 (See [27,30])**.** The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1(J,E)$, is defined as

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt;\ \xi > 0.$$

Provided the integral exists.

Example 2.6. Let $q > 0$. Then

$${}^H I_1^q \ln t = \frac{1}{\Gamma(2+q)} (\ln t)^{1+q}; \text{for a.e. } t \in [0,e].$$

Definition 2.7 (See [27,30])**.** Let $1 \le a < T,\ q > 0$, and $g \in L^1(J,E)$. Then

$$({}^H I_{a^+}^q g)(x) = \frac{1}{\Gamma(q)} \int_{a^+}^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

Remark 2.8. Let $g \in P_1(J,E)$. For every $\varphi \in E^*$, we have

$$\varphi({}^H I_1^q g)(x) = ({}^H I_1^q \varphi g)(x); \text{for a.e. } x \in J.$$

Analogously to the Riemann–Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way. Set

$$\delta = x\frac{d}{dx},\ n = [q] + 1,$$

where $[q]$ is the integer part of $q > 0$, and

$$AC^n_\delta := \{u : J \to E : \delta^{n-1}[u(x)] \in AC(J)\}.$$

Definition 2.9 (See [27, 30])**.** The Hadamard fractional derivative of order q applied to the function $w \in AC^n_\delta$ is defined as

$$({}^H D^q_1 w)(x) = \delta^n ({}^H I^{n-q}_1 w)(x).$$

Example 2.10. Let $0 < q < 1$. Then

$${}^H D^q_1 \ln t = \frac{1}{\Gamma(2-q)}(\ln t)^{1-q};\text{ for a.e. } t \in [0, e].$$

Definition 2.11 (See [27, 30])**.** Let $1 \leq a < T$ and $g \in L^1(J, E)$. Then

$$({}^H D^q_{a^+} w)(x) = \delta^n ({}^H I^{n-q}_{a^+} w)(x).$$

It has been proved (see e.g., Kilbas [29, Theorem 4.8]) that in the space $L^1(J, E)$, the Hadamard fractional derivative is the left–inverse operator to the Hadamard fractional integral, i.e.,

$$({}^H D^q_1)({}^H I^q_1 w)(x) = w(x).$$

From [30, Theorem 2.3], we have

$$({}^H I^q_1)({}^H D^q_1 w)(x) = w(x) - \frac{({}^H I^{1-q}_1 w)(1)}{\Gamma(q)}(\ln x)^{q-1}.$$

Corollary 2.12. *Let $h : J_0 \to E$ be a continuous function. A function $u \in L^1(J_0, E)$ is a solution of the equation*

$$({}^H D^q_1 u)(t) = h(t),$$

if and only if, u satisfies the Hadamard integral equation

$$u(t) = \frac{({}^H I^{1-q}_1 u)(1)}{\Gamma(q)}(\ln t)^{q-1} + ({}^H I^q_1 h)(t).$$

Lemma 2.13. *Let $h : J \to E$ be a continuous function. A function $u \in L^1(J, E)$ is solution of the fractional integral equation*

$$\begin{cases} u(t) = \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^HI_1^r h)(t);\ if\ t \in J_0, \\ u(t) = \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + \sum_{i=1}^{k} L_i(u(t_i^-)) \\ + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{r-1} \frac{h(s)}{s\Gamma(r)} ds \\ + \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{r-1} \frac{h(s)}{s\Gamma(r)} ds;\ if\ t \in J_k,\ k = 1, \dots, m, \end{cases} \tag{2.1}$$

if and only if u is a solution of the problem

$$\begin{cases} ({}^HD_{t_k}^r u)(t) = h(t);\ t \in J_k,\ k = 0, \dots, m, \\ \frac{(\ln t)^{r-1}}{\Gamma(r)}({}^HI_{t_k}^{1-r} u)(t_k^+) = u(t_k^-) + L_k(u(t_k^-));\ k = 1, \dots, m, \\ ({}^HI_1^{1-r} u)(t)|_{t=1} = \phi. \end{cases} \tag{2.2}$$

Proof. Assume u satisfies (2.2). If $t \in J_0$, then

$$({}^HD_1^r u)(t) = h(t).$$

Corollary 2.12 implies

$$u(t) = \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^HI_1^r h)(t).$$

If $t \in J_1$, then

$$({}^HD_{t_1}^r u)(t) = h(t).$$

Corollary 2.12 implies

$$\begin{aligned} u(t) &= \frac{({}^HI_{t_1}^{1-r} u)(t_1^+)}{\Gamma(r)}(\ln t)^{r-1} + ({}^HI_{t_1}^r h)(t) \\ &= L_1(u(t_1^-)) + u(t_1^-) + ({}^HI_{t_1}^r h)(t) \\ &= L_1(u(t_1^-)) + \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^HI_1^r h)(t_1) + ({}^HI_{t_1}^r h)(t). \end{aligned}$$

If $t \in J_2$, then

$$({}^HD_2^r u)(t) = h(t).$$

Corollary 2.12 implies

$$\begin{aligned} u(t) &= \frac{({}^H I_{t_2}^{1-r}u)(t_2^+)}{\Gamma(r)}(\ln t)^{r-1} + ({}^H I_{t_2}^r h)(t) \\ &= L_2(u(t_2^-)) + u(t_2^-) + ({}^H I_{t_2}^r h)(t) \\ &= L_2(u(t_2^-)) + L_1(u(t_1^-)) + \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} \\ &\quad + ({}^H I_1^r h)(t_1) + ({}^H I_{t_1}^r h)(t_2) + ({}^H I_{t_2}^r h)(t). \end{aligned}$$

If $t \in J_k$, then again from Corollary 2.12 we get (2.1). Conversely, assume that u satisfies the impulsive fractional integral equations (2.1). If $t \in J_0$, then $u(t) = \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^H I_1^r h)(t)$. Thus, $({}^H I_1^{1-r}u)(t)|_{t=1} = \phi$ and using the fact that ${}^H D_1^r$ is the left inverse of ${}^H I_1^r$ we get $({}^H D_1^r u)(t) = h(t)$.

Now, if $t \in J_k;\ k = 1, \ldots, m$, we get $({}^H D_{t_k}^r u)(t) = h(t)$. Also, we can easily show that

$$\frac{(\ln t)^{r-1}}{\Gamma(r)}({}^H I_{t_k}^{1-r}u)(t_k^+) = u(t_k^-) + L_k(u(t_k^-)).$$

Hence, if u satisfies the impulsive fractional integral equations (2.1) then we get (2.2). □

As a consequence; we have the following lemma.

Lemma 2.14. *Let $f(t,u,z) : J_k \times E \times E \to E;\ k = 0, \ldots, m$, be a continuous function. Then problem* (1.1) *is equivalent to the problem of the solution of the equation*

$$g(t) = f\left(t, \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^H I_{t_k}^r g)(t), g(t)\right),$$

and if $g(t) \in C(J_k);\ k = 0, \ldots, m$, is the solution of the above equation, then

$$\begin{cases} u(t) = \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^H I_1^r g)(t);\ \textit{if}\ t \in J_0, \\ u(t) = \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + \sum_{i=1}^{k}(L_i(({}^H I_{t_i}^{1-r}u)(t_i^-)) \\ + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln \frac{t_i}{s}\right)^{r-1}\frac{g(s)}{s\Gamma(r)}ds \\ + \int_{t_k}^{t}\left(\ln \frac{t}{s}\right)^{r-1}\frac{g(s)}{s\Gamma(r)}ds;\ \textit{if}\ t \in J_k,\ k = 1, \ldots, m. \end{cases}$$

Definition 2.15 (See [24])**.** Let E be a Banach space, Ω_E the bounded subsets of E and B_1 the unit ball of E. The De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \to [0, \infty)$ defined by

$\beta(X) = \inf\{\epsilon > 0 :$ there exists a weakly compact subset Ω of $E : X \subset \epsilon B_1 + \Omega\}$.

The De Blasi measure of weak noncompactness satisfies the following properties:

(a) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$,

(b) $\beta(A) = 0 \Leftrightarrow$ A is weakly relatively compact,

(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$,

(d) $\beta(\overline{A}^{\omega}) = \beta(A)$, ($\overline{A}^{\omega}$ denotes the weak closure of A),

(e) $\beta(A + B) \leq \beta(A) + \beta(B)$,

(f) $\beta(\lambda A) = |\lambda|\beta(A)$,

(g) $\beta(\mathrm{conv}(A)) = \beta(A)$,

(h) $\beta(\cup_{|\lambda| \leq h} \lambda A) = h\beta(A)$.

The next result follows directly from the Hahn–Banach theorem.

Proposition 2.16. *Let E be a normed space, and $x_0 \in E$ with $x_0 \neq 0$. Then, there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v : J \to E$ let us denote by

$$V(t) = \{v(t) : v \in V\};\ t \in J,$$

and

$$V(I) = \{v(t) : v \in V,\ t \in J\}.$$

Lemma 2.17 (See [26])**.** *Let $H \subset C$ be a bounded and equicontinuous. Then the function $t \to \beta(H(t)$ is continuous on J, and*

$$\beta_C(H) = \max_{t \in J} \beta(H(t)),$$

and

$$\beta\left(\int_J u(s)ds\right) \leq \int_J \beta(H(s))ds,$$

where $H(s) = \{u(s) : u \in H,\ s \in J\}$, and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of C.

For our purpose we will need the following fixed point theorem.

Theorem 2.18 (See [33])**.** *Let Q be a nonempty, closed, convex and equicontinuous subset of a metrizable locally convex vector space $C(J, E)$ such that $0 \in Q$. Suppose $T : Q \to Q$ is weakly-sequentially continuous. If the implication*

$$\overline{V} = \overline{\mathrm{conv}}(\{0\} \cup T(V)) \Rightarrow V \text{ is relatively weakly compact}, \tag{2.3}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

3 Existence Results

Let us start by defining what we mean by a weak solution of the problem (1.1).

Definition 3.1. By a weak solution of the problem (1.1) we mean a measurable function $u \in PC(J)$ that satisfies the condition $({}^H I_1^{1-r} u)(t)|_{t=1} = \phi$, and the equation $({}^H D_{t_k}^r u)(t) = f(t, u(t), ({}^H D_{t_k}^r u)(t))$ on $J_k;\ k = 0, \ldots, m$.

The following hypotheses will be used in the sequel.

(H_1) For a.e. $t \in J_k;\ k = 0, \ldots, m$, the functions $v \to f(t, v, \cdot)$ and $w \to f(t, \cdot, w)$ are weakly sequentially continuous.

(H_2) For a.e. $v, w \in E$, the function $t \to f(t, v, w)$ is Pettis integrable a.e. on $J_k;\ k = 0, \ldots, m$.

(H_3) There exists $p \in C(J_k, [0, \infty));\ k = 0, \ldots, m$ such that for all $\varphi \in E^*$, we have

$$|\varphi(f(t, u, v))| \leq \frac{p(t)\|\varphi\|}{1 + \|\varphi\| + \|u\|_E + \|v\|_E};\ \text{for a.e. } t \in J_k,\ \text{and each } u, v \in E.$$

(H_4) For each bounded and measurable set $B \subset E$ and for each $t \in J_k;\ k = 0, \ldots, m$, we have

$$\beta(f(t, B, {}^H D_1^r B) \leq (\ln t)^{1-r} p(t) \beta(B),$$

where ${}^H D_1^r B = \{{}^H D_1^r w : w \in B\}$.

(H_5) There exists a constant $l^* > 0$ such that for all $\varphi \in E^*$, we have

$$|\varphi(L_k(u))| \leq \frac{l^*\|\varphi\|}{1 + \|\varphi\| + \|u\|_E};\ \text{for a.e. } t \in J_k;\ k = 1, \ldots, m,\ \text{and each } u \in E.$$

Set

$$p^* = \sup_{t \in J} p(t),$$

Theorem 3.2. *Assume that the hypotheses* $(H_1) - (H_4)$ *hold. If*

$$L := m l^* (\ln T)^{1-r} + \frac{2p^* \ln T}{\Gamma(1+r)} < 1, \tag{3.1}$$

then the problem (1.1) *has at least one solution defined on* I.

Proof. Transform the problem (1.1) into a fixed point equation. Consider the operator $N : \mathcal{PC} \to \mathcal{PC}$ defined by

$$\begin{cases} (Nu)(t) = \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^H I_1^r g)(t); \text{ if } t \in J_0, \\ (Nu)(t) = \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + \sum_{i=1}^{k} L_i(u(t_i^-)) \\ + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{r-1} \frac{g(s)}{s\Gamma(r)} ds \\ + \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{r-1} \frac{g(s)}{s\Gamma(r)} ds; \text{ if } t \in J_k,\ k = 1, \dots, m, \end{cases} \tag{3.2}$$

where $g \in C(J_k);\ k = 0, \dots, m$, with

$$g(t) = f\left(t, \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^H I_{t_k}^r g)(t), g(t)\right).$$

First notice that, the hypotheses imply that $\left(\ln \frac{t_k}{s}\right)^{r-1} \frac{g(s)}{s}$; for all $t \in J_k,\ k = 0, \dots, m,$ is Pettis integrable, and for each $u \in C$, the function

$$t \mapsto f\left(t, \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^H I_{t_k}^r g)(t), g(t)\right) :\ k = 0, \dots, m,$$

is Pettis integrable over $J_k;\ k = 0, \dots, m$. Thus, the operator N is well defined. Let $R > 0$ be such that

$$R > ml^*(\ln T)^{1-r} + \frac{2p^* \ln T}{\Gamma(1+r)},$$

and consider the set

$$\begin{aligned} Q = \Big\{ u \in \mathcal{PC} : \|u\|_{\mathcal{PC}} \le R \text{ and } &\|(\ln x_2)^{1-r}u(x_2) - (\ln x_1)^{1-r}u(x_1)\|_E \\ &\le ml^* \left|(\ln x_2)^{1-r} - (\ln x_1)^{1-r}\right| + \frac{2p^*}{\Gamma(1+r)}(\ln T)^{1-r} \left|\ln \frac{x_2}{x_1}\right|^r \\ + \frac{2p^*}{\Gamma(r)} \int_1^{x_1} &\left|(\ln x_2)^{1-r}\left(\ln \frac{x_2}{s}\right)^{r-1} - (\ln x_1)^{1-r}\left(\ln \frac{x_1}{s}\right)^{r-1}\right| ds \Big\}. \end{aligned}$$

Clearly, the subset Q is closed, convex end equicontinuous. We shall show that the operator N satisfies all the assumptions of Theorem 2.18. The proof will be given in several steps.

Step 1. *N maps Q into itself.*
Let $u \in Q,\ t \in J_0$ and assume that $(Nu)(t) \neq 0$. Then there exists $\varphi \in E^*$ such that

$$\|(\ln t)^{1-r}(Nu)(t)\|_E = \varphi((\ln t)^{1-r}(Nu)(t)).$$

Thus

$$\|(\ln t)^{1-r}(Nu)(t)\|_E = \varphi\left(\frac{\phi}{\Gamma(r)} + \frac{(\ln t)^{1-r}}{\Gamma(r)}\int_1^t \left(\ln\frac{t}{s}\right)^{r-1}\frac{g(s)}{s}ds\right),$$

where $g \in C$ with

$$g(t) = f\left(t, \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^H I_1^r g)(t), g(t)\right).$$

Then

$$\begin{aligned}
\|(\ln t)^{1-r}(Nu)(t)\|_E &\le \frac{(\ln t)^{1-r}}{\Gamma(r)}\int_1^t \left(\ln\frac{t}{s}\right)^{r-1}\frac{|\varphi(g(s))|}{s}ds \\
&\le \frac{p^*(\ln T)^{1-r}}{\Gamma(r)}\int_1^t \left(\ln\frac{t}{s}\right)^{r-1}\frac{ds}{s} \\
&\le \frac{p^*\ln T}{\Gamma(1+r)} \\
&\le R.
\end{aligned}$$

Also, if $u \in Q,\ t \in J_k:\ k = 1, \ldots, m$, we get

$$\begin{aligned}
\|(\ln t)^{1-r}(Nu)(t)\|_E &\le \sum_{i=1}^{k}\varphi((\ln t)^{1-r}L_i(u(t_i^-))) \\
&+ (\ln T)^{1-r}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{r-1}\frac{\varphi(g(s))}{s\Gamma(r)}ds \\
&+ (\ln T)^{1-r}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{r-1}\frac{\varphi(g(s))}{s\Gamma(r)}ds \\
&\le ml^*(\ln T)^{1-r} + \frac{2p^*\ln T}{\Gamma(1+r)} \\
&\le R.
\end{aligned}$$

Next, let $x_1, x_2 \in J_0$ such that $1 \le x_1 < x_2 \le t_1$ and let $u \in Q$, with

$$(\ln x_2)^{1-r}(Nu)(x_2) - (\ln x_1)^{1-r}(Nu)(x_1) \neq 0.$$

Then there exists $\varphi \in E^*$ such that

$$\begin{aligned}
&\|(\ln x_2)^{1-r}(Nu)(x_2) - (\ln x_1)^{1-r}(Nu)(x_1)\|_E \\
&\qquad = \varphi((\ln x_2)^{1-r}(Nu)(x_2) - (\ln x_1)^{1-r}(Nu)(x_1))
\end{aligned}$$

and $\|\varphi\| = 1$. Then

$$\begin{aligned}
&\|(\ln x_2)^{1-r}(Nu)(x_2) - (\ln x_1)^{1-r}(Nu)(x_1)\|_E \\
&= \varphi((\ln x_2)^{1-r}(Nu)(x_2) - (\ln x_1)^{1-r}(Nu)(x_1)) \\
&\leq ml^* \left|(\ln x_2)^{1-r} - (\ln x_1)^{1-r}\right| \\
&+ \varphi\left((\ln x_2)^{1-r}\int_1^{x_2}\left(\ln\frac{x_2}{s}\right)^{r-1}\frac{g(s)}{s\Gamma(r)}ds - (\ln x_1)^{1-r}\int_1^{x_1}\left(\ln\frac{x_1}{s}\right)^{r-1}\frac{g(s)}{s\Gamma(r)}ds\right),
\end{aligned}$$

where $g \in C$ with

$$g(t) = f\left(t, \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^H I_1^r g)(t), g(t)\right).$$

Then

$$\begin{aligned}
&\|(\ln x_2)^{1-r}(Nu)(x_2) - (\ln x_1)^{1-r}(Nu)(x_1)\|_E \\
&\leq ml^* \left|(\ln x_2)^{1-r} - (\ln x_1)^{1-r}\right| \\
&\quad + (\ln x_2)^{1-r}\int_{x_1}^{x_2}\left|\ln\frac{x_2}{s}\right|^{r-1}\frac{|\varphi(g(s))|}{s\Gamma(r)}ds \\
&\quad + \int_1^{x_1}\left|(\ln x_2)^{1-r}\left(\ln\frac{x_2}{s}\right)^{r-1} - (\ln x_1)^{1-r}\left(\ln\frac{x_1}{s}\right)^{r-1}\right|\frac{|\varphi(g(s))|}{s\Gamma(r)}ds \\
&\leq ml^* \left|(\ln x_2)^{1-r} - (\ln x_1)^{1-r}\right| \\
&\quad + (\ln x_2)^{1-r}\int_{x_1}^{x_2}\left|\ln\frac{x_2}{s}\right|^{r-1}\frac{p(s)}{\Gamma(r)}ds \\
&\quad + \int_1^{x_1}\left|(\ln x_2)^{1-r}\left(\ln\frac{x_2}{s}\right)^{r-1} - (\ln x_1)^{1-r}\left(\ln\frac{x_1}{s}\right)^{r-1}\right|\frac{p(s)}{\Gamma(r)}ds.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\|(\ln x_2)^{1-r}(Nu)(x_2) - (\ln x_1)^{1-r}(Nu)(x_1)\|_E \\
&\leq ml^* \left|(\ln x_2)^{1-r} - (\ln x_1)^{1-r}\right| \\
&\quad + \frac{p^*}{\Gamma(1+r)}(\ln T)^{1-r}\left|\ln\frac{x_2}{x_1}\right|^r \\
&\quad + \frac{p^*}{\Gamma(r)}\int_1^{x_1}\left|(\ln x_2)^{1-r}\left(\ln\frac{x_2}{s}\right)^{r-1} - (\ln x_1)^{1-r}\left(\ln\frac{x_1}{s}\right)^{r-1}\right|ds.
\end{aligned}$$

Also, if we let $x_1, x_2 \in J_k;\ k = 1, \ldots, m$ such that $t_k \leq x_1 < x_2 \leq t_{k+1}$ and let $u \in Q$, we obtain

$$\begin{aligned}
&\|(\ln x_2)^{1-r}(Nu)(x_2) - (\ln x_1)^{1-r}(Nu)(x_1)\|_E \\
&\leq ml^* \left|(\ln x_2)^{1-r} - (\ln x_1)^{1-r}\right| + \frac{2p^*}{\Gamma(1+r)}(\ln T)^{1-r}\left|\ln\frac{x_2}{x_1}\right|^r \\
&\quad + \frac{2p^*}{\Gamma(r)}\int_1^{x_1}\left|(\ln x_2)^{1-r}\left(\ln\frac{x_2}{s}\right)^{r-1} - (\ln x_1)^{1-r}\left(\ln\frac{x_1}{s}\right)^{r-1}\right|ds.
\end{aligned}$$

Hence $N(Q) \subset Q$.

Step 2. *N is weakly-sequentially continuous.*
Let (u_n) be a sequence in Q and let $(u_n(t)) \to u(t)$ in (E, ω) for each $t \in J_k;\ k = 0, \ldots, m$. Fix $t \in J_k;\ k = 0, \ldots, m$, since f satisfies the assumption (H_1), we have $f(t, u_n(t), ^H D_{t_k} u_n(t))$ converges weakly uniformly to $f(t, u(t), ^H D_{t_k} u(t))$. Hence the Lebesgue dominated convergence theorem for Pettis integral implies $(Nu_n)(t)$ converges weakly uniformly to $(Nu)(t)$ in (E, ω), for each $t \in J_k;\ k = 0, \ldots, m$. Thus, $N(u_n) \to N(u)$. Hence, $N : Q \to Q$ is weakly-sequentially continuous.

Step 3. *The implication* (2.3) *holds.*
Let V be a subset of Q such that $\overline{V} = \overline{\text{conv}}(N(V) \cup \{0\})$. Obviously

$$V(t) \subset \overline{\text{conv}}(NV)(t)) \cup \{0\}),\ t \in J_k;\ k = 0, \ldots, m.$$

Further, as V is bounded and equicontinuous, by [23, Lemma 3] the function $t \to v(t) = \beta(V(t))$ is continuous on $J_k;\ k = 0, \ldots, m$. From $(H_3) - (H_5)$, Lemma 2.17 and the properties of the measure β, for any $t \in J_0$, we have

$$\begin{aligned}
(\ln t)^{1-r} v(t) &\leq \beta((\ln t)^{1-r}(NV)(t) \cup \{0\}) \\
&\leq \beta((\ln t)^{1-r}(NV)(t)) \\
&\leq \frac{(\ln T)^{1-r}}{\Gamma(r)} \int_1^t \left|\ln \frac{t}{s}\right|^{r-1} \frac{p(s)\beta(V(s))}{s} ds \\
&\leq \frac{(\ln T)^{1-r}}{\Gamma(r)} \int_1^t \left|\ln \frac{t}{s}\right|^{r-1} \frac{(\ln s)^{1-r} p(s) v(s)}{s} ds \\
&\leq \frac{p^* \ln T}{\Gamma(1+r)} \|v\|_C.
\end{aligned}$$

Thus

$$\|v\|_C \leq L\|v\|_C.$$

Also, for any $t \in J_k;\ k = 1, \ldots, m$, we get

$$\begin{aligned}
(\ln t)^{1-r} v(t) &\leq \beta((\ln t)^{1-r}(NV)(t) \cup \{0\}) \\
&\leq \beta((\ln t)^{1-r}(NV)(t)) \\
&\leq (\ln T)^{1-r} \sum_{i=1}^{k} l^* \beta(V(s)) \\
&+ (\ln T)^{1-r} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{r-1} \frac{p(s)\beta(V(s))}{s\Gamma(r)} ds \\
&+ (\ln T)^{1-r} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{r-1} \frac{p(s)\beta(V(s))}{s\Gamma(r)} ds \\
&\leq l^* (\ln T)^{1-r} \sum_{i=1}^{k} (\ln t)^{1-r} v(t)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad (\ln T)^{1-r} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s} \right)^{r-1} \frac{(\ln s)^{1-r} p(s) v(s)}{s \Gamma(r)} ds \\
&+ \quad (\ln T)^{1-r} \int_{t_k}^{t} \left(\ln \frac{t}{s} \right)^{r-1} \frac{(\ln s)^{1-r} p(s) v(s)}{s \Gamma(r)} ds \\
&\leq \quad \left(m l^* (\ln T)^{1-r} + \frac{2p^* \ln T}{\Gamma(1+r)} \right) \|v\|_C .
\end{aligned}
$$

Hence

$$\|v\|_C \leq L \|v\|_C .$$

From (3.1), we get $\|v\|_C = 0$, that is $v(t) = \beta(V(t)) = 0$, for each $t \in I$ and then by [31, Theorem 2], V is weakly relatively compact in $\mathcal{C}$. Applying now Theorem 2.18, we conclude that N has a fixed point which is a solution of the problem (1.1). □

4 An Example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n| .$$

Consider the problem of implicit impulsive Hadamard fractional differential equations of the form

$$
\begin{cases}
({}^H D_{t_k}^r u)(t) = f(t, u(t), ({}^H D_{t_k}^r u)(t)); \ t \in J_k, \ k = 0, \ldots, m, \\
\dfrac{(\ln t)^{r-1}}{\Gamma(r)} ({}^H I_{t_k}^{1-r} u)(t_k^+) = u(t_k^-) + L_k(u(t_k^-)); \ k = 1, \ldots, m, \\
({}^H I_1^{1-r} u)(t)|_{t=1} = 0,
\end{cases}
\tag{4.1}
$$

where $J = [1, e], \ r \in (0, 1], \ u = (u_1, u_2, \ldots, u_n, \ldots),$

$f = (f_1, f_2, \ldots, f_n, \ldots), \ {}^H D_{t_k}^r u = ({}^H D_{t_k}^r u_1, {}^H D_{t_k}^r u_2, \ldots, {}^H D_{t_k}^r u_n, \ldots); \ k = 0, \ldots, m,$

$$f_n(t, u(t), ({}^H D_{t_k}^r u)(t)) = \frac{ct^2}{1 + \|u(t)\|_E + \|{}^H D_{t_k}^r u(t)\|_E} \left(e^{-7} + \frac{1}{e^{t+5}} \right) u_n(t); \ t \in [1, e],$$

$$L_k(u(t_k^-)) = \frac{1}{3e^4 (1 + \|u(t_k^-)\|_E)}; \ k = 1, \ldots, m.$$

Clearly, the function f is continuous.
For each $u \in E$ and $t \in [1,e]$, we have

$$\|f(t,u(t),({}^H D^r_{t_k})(t))\|_E \leq \frac{ct^2}{1+\|u(t)\|_E+\|{}^H D^r_{t_k}u(t)\|_E}\left(e^{-7}+\frac{1}{e^{t+5}}\right),$$

and

$$\|L_k(u(t_k^-))\|_E \leq \frac{1}{3e^4(1+\|u(t_k^-)\|_E)}.$$

Hence, the hypothesis (H_3) is satisfied with $p^* = ce^{-4}$, and (H_5) is satisfied with $l^* = \frac{1}{3e^4}$.
We shall show that condition (3.1) holds with $T = e$. Indeed, if we assume, for instance, that the number of impulses $m = 3$, and $r = \frac{1}{2}$, then we have

$$ml^*(\ln T)^{1-r} + \frac{2p^* \ln T}{\Gamma(1+r)} = \frac{1}{e^4} + \frac{2c}{e^4\Gamma\left(\frac{3}{2}\right)} = \frac{9}{16} < 1.$$

A simple computations show that all conditions of Theorem 3.2 are satisfied. It follows that the problem (4.1) has at least one solution on $[1,e]$.

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