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Fixed Point Theorems for Expansive Mappings in js-Metric Space

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Abstract

In this paper, we introduce Kannan, Chatterjea, Zamfirscu and Rhodes type expansive mappings in the setting of generalized metric spaces (js-metric spaces). The results proved in the setting of generalized metric spaces generalizes the results in metric spaces, dislocated metric spaces, b-metric spaces, and modular spaces. We also illustrate our results with the help of certain examples.

Keywords: Fixed point, js-metric space, Kannan expansion, Zamfirscu expansion, Rhodes expansion.

1. INTRODUCTION

In the last few decades, fixed point theory is being one of the most interesting research subject in non linear analysis. In 1922, Banach [1] gave a new direction in research by introducing Banach Contraction Principle. After that Kannan [2],

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Chatterjea [3], Zamfirscu [4], Rhodes[5] gave generalisation of this result. In 1984, Wang [6] introduced the expansive mappings and proved fixed point results for them.

Now a days, the concept of standard metric spaces plays a role of fundamental tool in fixed point theory and also attract many researchers because of development of fixed point theory in standard metric spaces. In last few years, several generalization of standard metric spaces came into existence like b-metric spaces[7], dislocated metric spaces[8], modular spaces[9]. Recently in 2015, Jleli and Samet [10] introduced a new generalization of metric spaces (js-metric space).

In this paper, we establish some new results of fixed points by defining Kannan type, Zamfirscu, Rhodes fixed point theorem in generalized metric spaces (js-metric spaces) which recovers several topological spaces including dislocated metric spaces, b-metric spaces, modular spaces.

2. PRELIMINARIES

Definition 2.1[10] Let X be a nonempty and $\mathcal{D}: X \times X \to [0, \infty]$ be a given mapping. For every $p \in X$, let us define the set

$$C(\mathcal{D}, X, p) = \left\{\{p_n\} \subset X: \lim_{n\to\infty} \mathcal{D}(p_n, p) = 0\right\}.$$

Definition 2.2 [10] Let X be a non empty set and $\mathcal{D}: X \times X \to [0, \infty]$be a mapping. Then $(X, \mathcal{D})$ is a generalized metric on X if it satisfies the following conditions:

$(\mathcal{D}_1)$ for every $(p, q) \in X \times X$, we have $\mathcal{D}(p, q) = 0 \implies p = q$;

$(\mathcal{D}_2)$ for every $(p, q) \in X \times X$, we have $\mathcal{D}(p, q) = \mathcal{D}(q, p)$;

$(\mathcal{D}_3)$ there exists $C > 0$ such that if $(p, q) \in X \times X, \{p_n\} \in C(\mathcal{D}, X, p)$,

then $\mathcal{D}(p, q) \leq C limsup_{n\to\infty} \mathcal{D}(p_n, q)$.

In this case, the pair $(X, \mathcal{D})$ is a generalized metric space.

Remark 2.3 [10] If the set $C(\mathcal{D}, X, p)$ is empty for every $p \in X$, then $(X, \mathcal{D})$ is a generalized metric space if and only if $(\mathcal{D}_1)$ and $(\mathcal{D}_2)$ are satisfied.

Definition 2.4 [10] Let $(X, \mathcal{D})$ be a generalized metric space and $\{p_n\}$ be a sequence in X and $p \in X$. We say that $\{p_n\}$ $\mathcal{D}$-converges to p if $p \in C(\mathcal{D}, X, p)$.

Proposition 2.5 [10] Let $(X,\mathcal{D})$ be a generalized metric space and $\{p_n\}$ be a sequence in X and $(p,q) \in X \times X$. If $\{p_n\}$ $\mathcal{D}$-converges to p and $\{p_n\}$ $\mathcal{D}$-converges to q, then $p = q$.

Definition 2.6 [10] Let $(X,\mathcal{D})$be a generalized metric space and $\{p_n\}$ be a sequence in X. We say that $\{p_n\}$ is a $\mathcal{D}$-cauchy sequence if $\lim_{n\to\infty} \mathcal{D}(p_n, p_{n+m}) = 0$.

Definition 2.7 [10] Let $(X,\mathcal{D})$be a generalized metric space. It is said to be $\mathcal{D}$-complete if every Cauchy sequence in X is convergent to some element in X.

Definition 2.8 [10] Let $(X,\mathcal{D})$be a generalized metric space and $T: X \to X$ be a mapping. Let $k \in (0,1)$ then T is said to be k-contraction if $\mathcal{D}(T(p),T(q)) \leq k\mathcal{D}(p,q)$, for every $(p,q) \in X \times X$.

Definition 2.9 [10] Suppose that T is a k-contraction for some $k \in (0,1)$. Then any fixed point $u \in X$ of T satisfies $\mathcal{D}(u,u) < \infty \Longrightarrow D(u,u) = 0$.

Definition 2.10 [10] For every $p \in X$, let $\delta(\mathcal{D},T,p) = sup\left\{\mathcal{D}\left(T^i(p),T^j(p)\right): i,j \in \mathbb{N}\right\}$.

Definition 2.11 [10] Suppose that the following conditions hold:

(i) $(X,\mathcal{D})$is complete;
(ii) T is a k-contraction for some $k \in (0,1)$;
(iii) there exists $p_0 \in X$ such that $\delta(\mathcal{D},T,p_0) < \infty$.

Then $\{T^n(p_0)\}$ converges to $u \in X$, a fixed point of T. Moreover, if $v \in X$ is another fixed point of f such that $\mathcal{D}(u,v) < \infty$, then $u = v$.

Definition 2.12[10] Let $k \in (0,1)$, it is said to be k-quasicontraction if

$$\mathcal{D}(T(p),T(q)) \leq k\ max\{\mathcal{D}(p,q),\mathcal{D}(p,Tp),\mathcal{D}(q,Tq),\mathcal{D}(p,Tq),\mathcal{D}(q,Tp)\},$$

for every $(p,q) \in X \times X$.

Theorem 2.13[10] Suppose that the following conditions hold:

(i) $(X,\mathcal{D})$is complete;
(ii) T is a k-quasicontraction for some $k \in (0,1)$;
(iii) there exists $p_0 \in X$ such that $\delta(\mathcal{D},T,p_0) < \infty$.

Then $\{T^n(p_0)\}$ converges to $u \in X$, a fixed point of T. Moreover, if $v \in X$ is another fixed point of T such that $\mathcal{D}(u,v) < \infty$, then $u = v$.

3. MAIN RESULT

Definition 3.1 Let $(X,\mathcal{D})$ be a generalized metric space and $T: X \to X$ be a onto mapping. Let $k > 1$, then T is said to be k-expansion if

$\mathcal{D}(T(x),T(y)) \geq k\mathcal{D}(x,y)$, for every $(x,y) \in X \times X$.

Definition 3.2 Let $(X,\mathcal{D})$ be a generalized metric space and $T: X \to X$ be a onto mapping Let $k > 1$, it is said to be k-quasiexpansion if

$\mathcal{D}(T(x),T(y)) \geq$
$k\, max\{\mathcal{D}(x,y),\mathcal{D}(x,Tx),\mathcal{D}(y,Ty),\mathcal{D}(x,Ty),\mathcal{D}(y,Tx)\}$, for every $(x,y) \in X \times X$.

Definition 3.3 Let $(X,\mathcal{D})$ be a generalized metric space and $T: X \to X$ be a onto mapping. Let $k \geq \frac{1}{2}$ then T is said to be k-Kannan expansion if

$\mathcal{D}(T(x),T(y)) \geq k[\mathcal{D}(x,Tx) + \mathcal{D}(y,Ty)]$, for every $(x,y) \in X \times X$.

Definition 3.4 Let $(X,\mathcal{D})$be a generalized metric space and $T: X \to X$ be a mapping. Let $k \geq \frac{1}{2}$ then T is said to be k- Chatterjea expansion if

$\mathcal{D}(T(x),T(y)) \geq k[\mathcal{D}(x\,,Ty) + \mathcal{D}(y,Tx)]$, for every $(x,y) \in X \times X$.

Definition 3.5 Let $(X,\mathcal{D})$be a generalized metric space and $T: X \to X$ be a mapping. Let $k \in (0,1)$ then T is said to be k-Zamfirscu expansion if

$\mathcal{D}(T(x),T(y)) \geq k\max\left\{\mathcal{D}(x,y),\frac{\mathcal{D}(x,Tx)+\mathcal{D}(y,Ty)}{2},\frac{\mathcal{D}(x,Ty)+\mathcal{D}(y,Tx)}{2}\right\}$, for every $(x,y) \in X \times X$.

Definition 3.6 Let $(X,\mathcal{D})$be a generalized metric space and $T: X \to X$ be a mapping. Let $k \in (0,1)$ then T is said to be k-Rhodes expansion if

$\mathcal{D}(T(x),T(y)) \geq k\max\left\{\mathcal{D}(x,y),\frac{\mathcal{D}(x,Tx)+\mathcal{D}(y,Ty)}{2},\mathcal{D}(x.Ty),\mathcal{D}(y,Tx)\right\}$, for every $(x,y) \in X \times X$.

Theorem 3.7 Let $(X,\mathcal{D})$ is complete generalized metric space and $T: X \to X$ be a onto mapping which satisfies the following conditions :

(i) T is a k-expansion for some $k \in (0,1)$;

(ii) there exists $x_0 \in X$ such that $\lim_{n\to\infty} \frac{\delta(\mathcal{D},T,x_0)}{k^n} < \infty$.

Then $\{T(x_0)\}$ converges to $\omega \in X$, a fixed point of T. Moreover, if $\omega' \in X$ is another fixed point of T such that $\mathcal{D}(\omega, \omega') < \infty$, then $\omega = \omega'$.

Proof Let $n \in \mathbb{N}$ $(n \geq 1)$. Since T is a k-expansion, for all $i, j \in \mathbb{N}$,we have

$$\mathcal{D}\left(T^{n-1+i}(x_0), T^{n-1+j}(x_0)\right) \geq k\mathcal{D}\left(T^{n+i}(x_0), T^{n+j}(x_0)\right),$$

which implies that

$$\delta\big(\mathcal{D}, T, T^n(x_0)\big) \leq \frac{1}{k}\delta\big(\mathcal{D}, T, T^{n-1}(x_0)\big).$$

Then, for every $n \in \mathbb{N}$, we have

$$\delta\big(\mathcal{D}, T, T^n(x_0)\big) \leq \frac{1}{k^n}\delta(\mathcal{D}, T, x_0).$$

Using the above inequality, for every $n, m \in \mathbb{N}$, we have

$$\mathcal{D}\big(T^n(x_0), T^{n+m}(x_0)\big) \leq \delta\big(\mathcal{D}, T, T^n(x_0)\big) \leq \frac{1}{k^n}\delta(\mathcal{D}, T, x_0).$$

Since $\lim_{n\to\infty} \frac{\delta(\mathcal{D},T,x_0)}{k^n} < \infty$ and $k > 1$, we get

$$\lim_{n,m\to\infty} \mathcal{D}\big(T^n(x_0), T^{n+m}(x_0)\big) = 0,$$

which implies that $\{T^n(x_0)\}$ is a $\mathcal{D}$-cauchy sequence.

But $(X, \mathcal{D})$ is $\mathcal{D}$-complete so there exists some $u \in X$ such that $\{T^n(x_0)\}$ is a $\mathcal{D}$-convergent to u.

Since T is onto, so there exist $\omega \in X$, such that $\omega \in T^{-1}(u) \Longrightarrow T(\omega) = u$

But by condition (i), T is k-expansion, for all $n \in \mathbb{N}$, we have

$$\mathcal{D}(T^n(x_0), u) = \mathcal{D}\big(T^{n+1}(x_0), T(\omega)\big) \geq k\mathcal{D}(T^n(x_0), \omega).$$

Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} \mathcal{D}(T^n(x_0), \omega) = 0.$$

Then $\{T^n(x_0)\}$ is $\mathcal{D}$-convergent to ω. By the proposition 2.5 the uniqueness of the limit we get, $u = \omega$, Hence, u is a fixed point of T

Now, suppose that $v \in X$ is a fixed point of T such that $\mathcal{D}(u, v) < \infty$. Since T is a k-expansion, we have

$$\mathcal{D}(u, v) = \mathcal{D}\big(T(u), T(v)\big) \geq k\mathcal{D}(u, v),$$

By property $(\mathcal{D}_1)$, we get $u = v$.

Observe that we can replace condition (ii) in Theorem 3.2 by

(H) there exists $x_0 \in X$ such that $sup\{\mathcal{D}\big(x_0, T^r(x_0)\big)\} < \infty$.

Example 3.8 Let $X = [0,1]$ be a complete generalized metric space with $\mathcal{D} = min\{x, y\}$ with $k = \frac{3}{2}$. Define the function $T: X \to X$ such that $T(x) = 2x$

If $x, y \in [0,1]$, without lose of generality $x < y$ then

$$\mathcal{D}(Tx, Ty) = min\{Tx, Ty\} = min\{2x, 2y\} = 2min\{x, y\} = 2\mathcal{D}(x, y).$$

Clearly, T is an expansive mapping. Now, for every $y \in X$, there exists an $x = \frac{y}{2} \in X$, such that, $y = T(x)$. So, T is onto. Clearly for all $i, 0 \leq T^i(x) \leq 2^i$, which implies that $\lim_{n\to\infty} \frac{\delta(\mathcal{D},T,x_0)}{k^n} < \infty$. So the condition (ii) of the Theorem 3.7 is also satisfied. So all the conditions of theorem 3.7 is satisfied with unique fixed point is $x = 0$.

Corollary 3.9 Let $(X, \mathcal{D})$be a complete b-metric space and $T: X \to X$ be a mapping. Suppose that for some $k > 1$, we have

$$\mathcal{D}\big(T(x), T(y)\big) \geq k\mathcal{D}(x, y), \text{ for every } (x, y) \in X \times X.$$

If there exists $x_0 \in X$ such that $sup\left\{d\left(T^i(x_0), T^j(x_0)\right): i, j \in \mathbb{N}\right\} < \infty$.

Then the sequence $\{T^n(x_0)\}$ converges to a fixed point of T. Moreover, T has one and only one fixed point.

Corollary 3.10 Let $(X, \mathcal{D})$be a complete dislocated metric space and $T: X \to X$ be a mapping. Suppose that for some $k > 1$, we have

$$\mathcal{D}\big(T(x), T(y)\big) \geq k\mathcal{D}(x, y), \text{ for every } (x, y) \in X \times X.$$

If there exists $x_0 \in X$ such that $sup\left\{d\left(T^i(x_0), T^j(x_0)\right): i, j \in \mathbb{N}\right\} < \infty$.

Then the sequence $\{T^n(x_0)\}$ converges to a fixed point of T. Moreover, T has one and only one fixed point.

Theorem 3.11 Let $(X, \mathcal{D})$ is complete generalized metric space and $T: X \to X$ be an onto mapping which satisfies following conditions:

(i) T is a k-quasi expansion for some $k \geq 1$;

(ii) there exists $x_0 \in X$ such that $\lim\limits_{n\to\infty} \frac{\delta(\mathcal{D},T,x_0)}{k^n} < \infty$.

Then $\{T^n(x_0)\}$ converges to $u \in X$, a fixed point of T. Moreover, if $v \in X$ is another fixed point of T such that $\mathcal{D}(u, v) < \infty$, then $u = v$.

Proof Let $n \in \mathbb{N}$ $(n \geq 1)$. Since T is a k-quasi expansion, for all $i, j \in \mathbb{N}$,we have

$$\mathcal{D}\left(T^{n-1+i}(x_0), T^{n-1+j}(x_0)\right) \geq k\, max \left\{\begin{matrix} \mathcal{D}\left(T^{n+i}(x_0), T^{n+j}(x_0)\right), \\ \mathcal{D}\left(T^{n+i}(x_0), T^{n-1+i}(x_0)\right), \\ \mathcal{D}\left(T^{n-1+i}(x_0), T^{n+j}(x_0)\right), \\ \mathcal{D}\left(T^{n+j}(x_0), T^{n-1+j}(x_0)\right), \\ \mathcal{D}\left(T^{n+i}(x_0), T^{n-1+j}(x_0)\right) \end{matrix}\right\}.$$

which implies that

$$\delta\big(\mathcal{D}, T, T^n(x_0)\big) \geq k\big[\delta\big(\mathcal{D}, T, T^{n-1}(x_0)\big)\big].$$

Then, for every $n \geq 1$, we have

$$\delta\big(\mathcal{D}, T, T^n(x_0)\big) \leq \frac{1}{k^n}\delta(\mathcal{D}, T, x_0).$$

Using the above inequality, for every $n, m \in \mathbb{N}$, we have

$$\mathcal{D}\big(T^n(x_0), T^{n+m}(x_0)\big) \leq \delta\big(\mathcal{D}, T, T^n(x_0)\big) \leq \frac{1}{k^n}\delta(\mathcal{D}, T, x_0).$$

Since $\lim\limits_{n\to\infty} \frac{\delta(\mathcal{D},T,x_0)}{k^n} < \infty$.and $k \geq 1$, we get

$$\lim_{n,m\to\infty} \mathcal{D}\big(T^n(x_0), T^{n+m}(x_0)\big) = 0,$$

which implies that $\{T^n(x_0)\}$ is a $\mathcal{D}$-cauchy sequence.

By condition (i), $(X,\mathcal{D})$ is $\mathcal{D}$-complete, there exists some $u \in X$ such that $\{T^n(x_0)\}$ is a $\mathcal{D}$-convergent to u.

But $(X,\mathcal{D})$ is $\mathcal{D}$-complete so there exists some $u \in X$ such that $\{T^n(x_0)\}$ is a $\mathcal{D}$-convergent to u.

Since T is onto, so there exist $\omega \in X$, such that $\omega \in T^{-1}(u) \Longrightarrow T(\omega) = u$

But by condition (i), T is k-quasi expansion, for all $n \in \mathbb{N}$, we have

$$\mathcal{D}(T^n(x_0),u) = \mathcal{D}\big(T^{n+1}(x_0),T(\omega)\big) \geq$$
$$kmax\begin{Bmatrix} \mathcal{D}(T^n(x_0),\omega),\mathcal{D}\big(T^n(x_0),T^{n+1}(x_0)\big), \\ \mathcal{D}\big(\omega,T(\omega)\big),\mathcal{D}\big(T^n(x_0),T(\omega)\big), \\ \mathcal{D}\big(\omega,T^{n+1}(x_0)\big) \end{Bmatrix}.$$

Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} \mathcal{D}(T^n(x_0),\omega) \geq k\mathcal{D}(\omega,u),$$

$$\mathcal{D}(\omega,u) \geq k\mathcal{D}(\omega,u).$$

Implies that $u = \omega$, Hence, u is a fixed point of T.

Remark 3.12 Using theorem 3.11 we can prove fixed point results for k-quasiexpansion in b-metric spaces, dislocated spaces and modular spaces.

Theorem 3.13 Let $(X,\mathcal{D})$ is complete generalized metric space and $T: X \to X$ be an onto mapping which satisfies following conditions:

(i) T is a k-Kannan expansion for some $k \geq \frac{1`}{2}$;

(ii) there exists $x_0 \in X$ such that $\lim_{n\to\infty} \frac{\delta(\mathcal{D},T,x_0)}{k^n} < \infty$.

Then $\{T^n(x_0)\}$ converges to $u \in X$, a fixed point of T Moreover, if $v \in X$ is another fixed point of T such that $\mathcal{D}(u,v) < \infty$, then $u = v$.

Proof Let $n \in \mathbb{N}$ $(n \geq 1)$. Since T is a k-Kannan expansion, for all $i,j \in \mathbb{N}$,we have

$$\mathcal{D}\left(T^{n-1+i}(x_0),T^{n-1+j}(x_0)\right) \geq k\ \left[\mathcal{D}\left(T^{n+i}(x_0),T^{n-1+i}(x_0)\right) + \mathcal{D}\left(T^{n+j}(x_0),T^{n-1+j}(x_0)\right)\right].$$

which implies that

$$\delta(\mathcal{D},T,T^n(x_0)) \geq k[\delta(\mathcal{D},T,T^{n-1}(x_0)) + \delta(D,T,T^{n-1}(x_0))] \geq k[2\delta(\mathcal{D},T,x_0)].$$

Then, for every $n \geq 1$, we have

$$\delta(\mathcal{D},T,T^n(x_0)) \leq \frac{1}{(2k)^n}\delta(\mathcal{D},T,x_0).$$

Using the above inequality, for every $n,m \in \mathbb{N}$, we have

$$\mathcal{D}(T^n(x_0),T^{n+m}(x_0)) \leq \delta(\mathcal{D},T,T^n(x_0)) \leq \frac{1}{(2k)^n}\delta(\mathcal{D},T,x_0).$$

Since $\lim_{n\to\infty}\frac{\delta(\mathcal{D},T,x_0)}{k^n} < \infty$.and $k \geq \frac{1}{2}$, we get

$$\lim_{n,m\to\infty}\mathcal{D}(T^n(x_0),T^{n+m}(x_0)) = 0,$$

which implies that $\{T^n(x_0)\}$ is a $\mathcal{D}$-cauchy sequence.

By condition (i), $(X,\mathcal{D})$ is $\mathcal{D}$-complete, there exists some $u \in X$ such that $\{T^n(x_0)\}$ is a $\mathcal{D}$-convergent to u.

But $(X,\mathcal{D})$ is $\mathcal{D}$-complete so there exists some $u \in X$ such that $\{T^n(x_0)\}$ is a $\mathcal{D}$-convergent to u.

Since T is onto, so there exist $\omega \in X$, such that $\omega \in T^{-1}(u) \Rightarrow T(\omega) = u$

But by condition (i), T is k-Kannan expansion, for all $n \in \mathbb{N}$, we have

$$\mathcal{D}(T^n(x_0),u) = \mathcal{D}(T^{n+1}(x_0),T(\omega)) \geq k[\mathcal{D}(T^n(x_0),T^{n+1}(x_0)) + \mathcal{D}(\omega,T(\omega))].$$

Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty}\mathcal{D}(T^n(x_0),\omega) \geq k\mathcal{D}(\omega,u),$$

$$\mathcal{D}(\omega,u) \geq k\mathcal{D}(\omega,u).$$

Implies that $u = \omega$, Hence, u is a fixed point of T.

Remark 3.14 Using theorem 3.13 we can prove fixed point results for k-quasiexpansion in b-metric spaces, dislocated spaces and modular spaces.

Example 3.15 Let $X = [0,2]$ be a complete generalized metric space with $\mathcal{D} = |x-y|$ with $k = \frac{1}{2}$. Define the function $T: X \to X$ such that $T(x) = \begin{cases} 2x & x < 1 \\ 2x-1 & x \geq 1 \end{cases}$

Clearly T satisfies the condition

$$\mathcal{D}\big(T(x), T(y)\big) \geq k[\mathcal{D}(x, Tx) + \mathcal{D}(y, Ty)].$$

Now, for every $y \in X$, there exists an $x = \begin{cases} \frac{y}{2} & for\ y < 2 \\ \frac{y+1}{2} & for\ y \geq 2 \end{cases} \in X$, such that, $y = T(x)$.

So, T is onto. Clearly for all i when $x < 1$, $0 \leq T^i(x) \leq 2^i$ and $0 \leq T^i(x) \leq 2^i - 1$ when $x \geq 1$ which implies that $\lim_{n\to\infty} \frac{\delta(\mathcal{D},T,x_0)}{k^n} < \infty$. So the condition (ii) of the Theorem 3.7 is also satisfied. So all the conditions of theorem 3.13 is satisfied with two fixed points $x = 0$ and $x = 1$.

Theorem 3.16 Let $(X, \mathcal{D})$ is complete generalized metric space and $T: X \to X$ be an onto mapping which satisfies following conditions:

(i) T is a k- Chatterjea expansion for some $k \geq \frac{1}{2}$;

(ii) there exists $x_0 \in X$ such that $\lim_{n\to\infty} \frac{\delta(\mathcal{D},T,x_0)}{k^n} < \infty$.

Then $\{T^n(x_0)\}$ converges to $u \in X$, a fixed point of T. Moreover, if $v \in X$ is another fixed point of T such that $\mathcal{D}(u, v) < \infty$, then $u = v$.

Proof Let $n \in \mathbb{N}$ $(n \geq 1)$. Since T is a k- Chatterjea expansion, for all $i, j \in \mathbb{N}$,we have

$$\mathcal{D}\left(T^{n-1+i}(x_0), T^{n-1+j}(x_0)\right) \geq k\ \left[\mathcal{D}\left(T^{n+i}(x_0), T^{n-1+j}(x_0)\right) + \mathcal{D}\left(T^{n+j}(x_0), T^{n-1+i}(x_0)\right)\right].$$

which implies that

$$\delta\big(\mathcal{D}, T, T^n(x_0)\big) \geq k\big[\delta\big(\mathcal{D}, T, T^{n-1}(x_0)\big) + \delta\big(D, T, T^{n-1}(x_0)\big)\big] \geq k[2\delta(\mathcal{D}, T, x_0)].$$

Then, for every $n \geq 1$, we have

$$\delta\big(\mathcal{D}, T, T^n(x_0)\big) \leq \frac{1}{(2k)^n} \delta(\mathcal{D}, T, x_0).$$

Using the above inequality, for every $n, m \in \mathbb{N}$, we have

$$\mathcal{D}\big(T^n(x_0), T^{n+m}(x_0)\big) \leq \delta\big(\mathcal{D}, T, T^n(x_0)\big) \leq \frac{1}{(2k)^n}\delta(\mathcal{D}, T, x_0).$$

Since $\lim\limits_{n\to\infty} \frac{\delta(\mathcal{D},T,x_0)}{k^n} < \infty$.and $k \geq \frac{1}{2}$, we get

$$\lim_{n,m\to\infty} \mathcal{D}\big(T^n(x_0), T^{n+m}(x_0)\big) = 0,$$

which implies that $\{T^n(x_0)\}$ is a $\mathcal{D}$-cauchy sequence.

By condition (i), $(X, \mathcal{D})$ is $\mathcal{D}$-complete, there exists some $u \in X$ such that $\{T^n(x_0)\}$ is a $\mathcal{D}$-convergent to u.

But $(X, \mathcal{D})$ is $\mathcal{D}$-complete so there exists some $u \in X$ such that $\{T^n(x_0)\}$ is a $\mathcal{D}$-convergent to u.

Since T is onto, so there exist $\omega \in X$, such that $\omega \in T^{-1}(u) \Longrightarrow T(\omega) = u$

But by condition (i), T is k- Chatterjea expansion, for all $n \in \mathbb{N}$, we have

$$\mathcal{D}(T^n(x_0), u) = \mathcal{D}\big(T^{n+1}(x_0), T(\omega)\big) \geq k\big[\mathcal{D}\big(T^n(x_0), T(\omega)\big) + \mathcal{D}\big(\omega, T^{n+1}(x_0)\big)\big].$$

Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} \mathcal{D}(T^n(x_0), \omega) \geq k[\mathcal{D}(u, \omega) + \mathcal{D}(\omega, u)],$$

$\mathcal{D}(\omega, u) \geq 2k\mathcal{D}(\omega, u)$. Since $k \geq \frac{1}{2}$

Which implies that $u = \omega$, Hence, u is a fixed point of T.

Theorem 3.17 Let $(X, \mathcal{D})$ is complete generalized metric space and $T: X \to X$ be an onto mapping which satisfies following conditions:

(i) T is a k- zamfirscu expansion for some $k \geq 1$;

(ii) there exists $x_0 \in X$ such that $\lim\limits_{n\to\infty} \frac{\delta(\mathcal{D},T,x_0)}{k^n} < \infty$.

Then $\{T^n(x_0)\}$ converges to $u \in X$, a fixed point of T. Moreover, if $v \in X$ is another fixed point of T such that $\mathcal{D}(u, v) < \infty$, then $u = v$.

Proof Let $n \in \mathbb{N}$ $(n \geq 1)$. Since T is a k- zamfirscu expansion, for all $i, j \in \mathbb{N}$,we have

$$\mathcal{D}\left(T^{n-1+i}(x_0), T^{n-1+j}(x_0)\right) \geq$$
$$k\, max \left\{ \begin{array}{c} \mathcal{D}\left(T^{n+i}(x_0), T^{n+j}(x_0)\right), \\ \frac{\mathcal{D}\left(T^{n+i}(x_0), T^{n-1+i}(x_0)\right) + \mathcal{D}\left(T^{n+j}(x_0), T^{n-1+j}(x_0)\right)}{2}, \\ \frac{\mathcal{D}\left(T^{n+i}(x_0), T^{n-1+j}(x_0)\right) + \mathcal{D}\left(T^{n+j}(x_0), T^{n-1+i}(x_0)\right)}{2} \end{array} \right\};$$

which implies that

$$\delta\big(\mathcal{D}, T, T^n(x_0)\big) \geq k\big[\delta\big(\mathcal{D}, T, T^{n-1}(x_0)\big)\big].$$

Then, for every $n \geq 1$, we have

$$\delta\big(\mathcal{D}, T, T^n(x_0)\big) \leq \frac{1}{k^n} \delta(\mathcal{D}, T, x_0).$$

Using the above inequality, for every $n, m \in \mathbb{N}$, we have

$$\mathcal{D}\big(T^n(x_0), T^{n+m}(x_0)\big) \leq \delta\big(\mathcal{D}, T, T^n(x_0)\big) \leq \frac{1}{k^n} \delta(\mathcal{D}, T, x_0).$$

Since $\lim_{n\to\infty} \frac{\delta(\mathcal{D}, T, x_0)}{k^n} < \infty$ and $k \geq 1$, we get

$$\lim_{n,m\to\infty} \mathcal{D}\big(T^n(x_0), T^{n+m}(x_0)\big) = 0,$$

which implies that $\{T^n(x_0)\}$ is a $\mathcal{D}$-cauchy sequence.

By condition (i), $(X, \mathcal{D})$ is $\mathcal{D}$-complete, there exists some $u \in X$ such that $\{T^n(x_0)\}$ is a $\mathcal{D}$-convergent to u.

But $(X, \mathcal{D})$ is $\mathcal{D}$-complete so there exists some $u \in X$ such that $\{T^n(x_0)\}$ is a $\mathcal{D}$-convergent to u.

Since T is onto, so there exist $\omega \in X$, such that $\omega \in T^{-1}(u) \Longrightarrow T(\omega) = u$

But by condition (ii), T is k- zamfirscu expansion, for all $n \in \mathbb{N}$, we have

$$\mathcal{D}(T^n(x_0), u) = \mathcal{D}\big(T^{n+1}(x_0), T(\omega)\big) \geq kmax \left\{ \begin{array}{c} \mathcal{D}(T^n(x_0), \omega), \\ \frac{\mathcal{D}\left(T^n(x_0), T^{n+1}(x_0)\right) + \mathcal{D}\left(\omega, T(\omega)\right)}{2}, \\ \frac{\mathcal{D}\left(T^n(x_0), T(\omega)\right) + \mathcal{D}\left(\omega, T^{n+1}(x_0)\right)}{2} \end{array} \right\}.$$

Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} \mathcal{D}(T^n(x_0), \omega) \geq k\mathcal{D}(\omega, u),$$

$$\mathcal{D}(\omega, u) \geq k\mathcal{D}(\omega, u).$$

Implies that $u = \omega$, Hence, u is a fixed point of T.

Remark 3.18 Using theorem 3.17, we can prove fixed point results for k- zamfirscu expansion in b-metric spaces, dislocated spaces and modular spaces.

Theorem 3.19 Let $(X, \mathcal{D})$ is complete generalized metric space and $T: X \to X$ be an onto mapping which satisfies following conditions:

(i) T is a k-Rhodes expansion for some $k \geq 1$;

(ii) there exists $x_0 \in X$ such that $\lim_{n\to\infty} \frac{\delta(\mathcal{D},T,x_0)}{k^n} < \infty$.

Then $\{T^n(x_0)\}$ converges to $u \in X$, a fixed point of T. Moreover, if $v \in X$ is another fixed point of T such that $\mathcal{D}(u, v) < \infty$, then $u = v$.

Proof Let $n \in \mathbb{N}$ $(n \geq 1)$. Since T is a k- Rhodes expansion, for all $i, j \in \mathbb{N}$,we have

$$\mathcal{D}\left(T^{n-1+i}(x_0), T^{n-1+j}(x_0)\right) \geq$$

$$k\, max \left\{ \begin{array}{c} \mathcal{D}\left(T^{n+i}(x_0), T^{n+j}(x_0)\right), \\ \frac{\mathcal{D}\left(T^{n+i}(x_0), T^{n-1+i}(x_0)\right) + \mathcal{D}\left(T^{n+j}(x_0), T^{n-1+j}(x_0)\right)}{2}, \\ \mathcal{D}\left(T^{n+j}(x_0), T^{n-1+i}(x_0)\right), \\ \mathcal{D}\left(T^{n+i}(x_0), T^{n-1+j}(x_0)\right) \end{array} \right\}.$$

which implies that

$$\delta\left(\mathcal{D}, T, T^n(x_0)\right) \geq k\left[\delta\left(\mathcal{D}, T, T^{n-1}(x_0)\right)\right].$$

Then, for every $n \geq 1$, we have

$$\delta\left(\mathcal{D}, T, T^n(x_0)\right) \leq \frac{1}{k^n} \delta(\mathcal{D}, T, x_0).$$

Using the above inequality, for every $n, m \in \mathbb{N}$, we have

$$\mathcal{D}\left(T^n(x_0), T^{n+m}(x_0)\right) \leq \delta\left(\mathcal{D}, T, T^n(x_0)\right) \leq \frac{1}{k^n} \delta(\mathcal{D}, T, x_0).$$

Since $\lim_{n\to\infty} \frac{\delta(\mathcal{D},T,x_0)}{k^n} < \infty$.and $k \geq 1$, we get

$$\lim_{n,m\to\infty} \mathcal{D}\big(T^n(x_0), T^{n+m}(x_0)\big) = 0,$$

which implies that $\{T^n(x_0)\}$ is a $\mathcal{D}$-cauchy sequence.

By condition (i), $(X,\mathcal{D})$ is $\mathcal{D}$-complete, there exists some $u \in X$ such that $\{T^n(x_0)\}$ is a $\mathcal{D}$-convergent to u.

But $(X,\mathcal{D})$ is $\mathcal{D}$-complete so there exists some $u \in X$ such that $\{T^n(x_0)\}$ is a $\mathcal{D}$-convergent to u.

Since T is onto, so there exist $\omega \in X$, such that $\omega \in T^{-1}(u) \Longrightarrow T(\omega) = u$

But by condition (ii), T is k- Rhodes expansion, for all $n \in \mathbb{N}$, we have

$$\mathcal{D}(T^n(x_0), u) = \mathcal{D}\big(T^{n+1}(x_0), T(\omega)\big) \geq$$
$$kmax \left\{ \begin{matrix} \mathcal{D}(T^n(x_0), \omega), \frac{\mathcal{D}\left(T^n(x_0), T^{n+1}(x_0)\right) + \mathcal{D}(\omega, T(\omega))}{2}, \\ \mathcal{D}\big(T^n(x_0), T(\omega)\big), \mathcal{D}\big(\omega, T^{n+1}(x_0)\big) \end{matrix} \right\}.$$

Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} \mathcal{D}(T^n(x_0), \omega) \geq k\mathcal{D}(\omega, u),$$

$$\mathcal{D}(\omega, u) \geq k\mathcal{D}(\omega, u).$$

Implies that $u = \omega$, Hence, u is a fixed point of T.

Remark 3.20 Using theorem 3.19 we can prove fixed point results for k- Rhodes expansion in b-metric spaces, dislocated spaces and modular spaces.

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