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Common Fixed Points for Multivalued Mappings

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ABSTRACT

In this paper we have extended the result of Sayyed [10]. The result is a generalized concept of commuting and compatible mappings under some conditions and corresponding result of Beg and Azam [1], Falset et. al [2], Jungck [3, 4], Kaneko [5] Nadler [7], Reich [8], Yadav et. al [13], Wang and Song [12] and many others.

Key words and Phrases: Hausdorff metric, Multivalued mappings, compatible mapping, complete metric space and coincidence point.

AMS (2010) subject classifications: primary 54H25; Secondary 47H10

INTRODUCTION

Banach obtained a fixed point theorem for contraction mapping, appearance of the celebrated banach contraction principle, several generalizations of this theorem in the setting of point mappings have been obtained. Nadler [7] was the first to extend Banach contraction principle to multivalued contracting mapping.

Rhoades [9] gave a complete and comparison of various definitions of contraction mapping and also survey of the subject. In this direction sayyed et. al [11], Lateef et. al. [6] proved a common fixed point theorem for multivalued and compatible maps.

The purpose of this paper to further demonstrate the effectiveness of the compatible map concept as a mean of multivalued and single valued maps satisfying a contractive type condition.

2. PRELIMINARIES

Let (X, d) be a metric space and let $CB(X)$ denote the family of all non-empty bounded closed subsets of X. For $A, B \in CB(X)$, let $H(A, B)$ denote the distance between A and B in Hausdorff metric, that is

$H(A, B) = \inf E_{AB}$

Where
$E_{AB} = \{\varepsilon > 0: A \subset N(\varepsilon, B), B \subset N(\varepsilon, A)\}$
$N(\varepsilon, A) = \{x : d(x, A) < \varepsilon\}$.

A point x is said to be a fixed point of a single valued mapping $f : X \to X$ (multivalued mapping $T : X \to CB(X)$) provided $x = fx$ ($x \in Tx$). The point x is called coincidence point of f and T, if $fx \in Tx$. If each element of X is a coincidence point of f and T, then f is called a selection of T.

Let $T : X \to CB(X)$ be a mapping, then $C_T = \{f : X \to X : TX \subset fX$ and $(\forall x \in X)$ $(f\,Tx = T\,fx)\}$. T and f are said to be commuting mappings if for each $x \in X$, $f(Tx) = fTx = Tfx = T(fx)$.

Lemma 2. 1: {Beg [1, Lemma 2. 1] }. Let S, T be two multivalued mappings of X into $CB(X)$. Let $x_0, x_1 \in X$. Then for each $y \in T(x_1)$ one has

$$d(y, Sx_0) \le H(Tx_1, Sx_0).$$

Theorem 2. 2: Let S, T be two mappings from a complete metric space X into $CB(X)$ and let $f \in C_S \cap C_T$ be continuous mapping. Suppose that for all $x, y \in X$,

$$\begin{aligned}[H(Sx,Ty)]^2 &\le \alpha[d(fx,Sx)d(fy,Ty) + d(fx,Ty)d(fy,Sx)] \\ &+ \beta[d(fx,Sx)d(fy,Sx) + d(fy,Ty)d(fx,Ty)] \\ &+ \gamma[d(fx,Sx) + d(fy,Ty)]H(Sx,Ty) \qquad \text{.... (1)}\end{aligned}$$

Where $\alpha, \beta, \gamma \ge 0$ and $0 \le \alpha + 2\beta + 2\gamma < 1$. Then there exists a common coincidence point of f and T and f and S.

Proof: Define $M = \dfrac{\alpha + \beta + \gamma}{1 - \beta - \gamma}$. Let x_0 be an arbitrary, but fixed element of X. We shall construct two sequences $\{x_n\}$ and $\{y_n\}$ as follows.

Let $x_1 \in X$ be such that $y_1 = fx_1 \in Sx_0$, using the definition of Hausdorff metric and fact that $Tx \subset fx$, we may choose $x_2 \in X$ such that $y_2 = fx_2 \in Tx_1$ and $d(y_1, y_2) = d(fx_1, fx_2) \le H(Sx_0, Tx_1) + (\alpha + \beta + \gamma)$.

Since $S(X) \subset f(x)$, we may choose $x_3 \in X$ such that $y_3 = fx_3 \in Sx_2$ and $d(y_2, y_3) = d(fx_2, fx_3) \le H(Tx_1, Sx_2) + \dfrac{(\alpha + \beta + \gamma)^2}{1 - \beta - \gamma}$.

By induction, we produce two sequence of points of X such that

$$y_{2k+1} = fx_{2k+1} \in Sx_{2k},$$
$$y_{2k+2} = fx_{2k+2} \in Tx_{2k+1}, \qquad (2)$$

Where k is any positive integer. Further more

$$d(y_{2k+1},y_{2k+2})=d(fx_{2k+1},fx_{2k+2})$$

$$\leq H(Sx_{2k},Tx_{2k+1})+\frac{(\alpha+\beta+\gamma)^{2k+1}}{(1-\beta-\gamma)^{2k}}$$

$$d(y_{2k+2},y_{2k+3})=d(fx_{2k+2},fx_{2k+3})$$

$$\leq H(Tx_{2k+1},Sx_{2k+2})+\frac{(\alpha+\beta+\gamma)^{2k+2}}{(1-\beta-\gamma)^{2k+1}}$$

Hence

$$[d(fx_{2k+1},fx_{2k+2})]^2<\alpha[d(fx_{2k},Sx_{2k})]d(fx_{2k+1},Tx_{2k+1})$$

$$+d(fx_{2k},Tx_{2k+1})d(fx_{2k+1},Sx_{2k})]$$

$$+\beta[d(fx_{2k},Sx_{2k})d(fx_{2k+1},Sx_{2k})$$

$$+d(fx_{2k+1},Tx_{2k+1})d(fx_{2k},Tx_{2k+1})]$$

$$+\gamma\ [d\ (fx_{2k},Sx_{2k})+d\ (fx_{2k+1},Tx_{2k+1})]\ d\ (fx_{2k+1},fx_{2k+2})$$

$$+\frac{(\alpha+\beta+\gamma)^{2k+1}}{(1-\beta-\gamma)^{2k}}$$

$$d(fx_{2k+1},fx_{2k+2})<(\alpha+\beta+\gamma)d(fx_{2k},fx_{2k+1})+(\beta+\gamma)d(fx_{2k+1},fx_{2k+2})]$$

$$+\frac{(\alpha+\beta+\gamma)^{2k+1}}{(1-\beta-\gamma)^{2k}}$$

$$d(fx_{2k+1},fx_{2k+2})\leq\frac{(\alpha+\beta+\gamma)}{(1-\beta-\gamma)}d(fx_{2k},fx_{2k+1})+\frac{(\alpha+\beta+\gamma)^{2k+1}}{(1-\beta-\gamma)^{2k+1}}$$

Therefore,

$$d(fx_{2k+1},fx_{2k+2})\leq Md(fx_{2k},fx_{2k+1})+M^{2k+1}$$

Similarly,

$$d(fx_{2k},fx_{2k+1})\leq Hd(Tx_{2k},Sx_{2k})+\frac{(\alpha+\beta+\gamma)^{2k}}{(1-\beta-\gamma)^{2k-1}}$$

Therefore,

$$d(fx_{2k},fx_{2k+1})\leq Md(fx_{2k-1},fx_{2k})+M^{2k}$$

It further implies that

$$d(y_n,y_{n+1})\leq Md(y_{n-1},y_n)+M^n$$

$$\leq M^{n-1}d(y_1,y_2)+(n-1)M^n$$

$$\leq M^{n-1}d(fx_1,fx_2)+(n-1)M^n$$

for $p \geq 1$, we have

$$d(y_{n+1}, y_{n+p+1}) \leq d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + ... + d(y_{n+p}, y_{n+p+1})$$
$$\leq \{M^n d(fx_1, fx_2) + nM^{n+1}\}$$
$$+\{M^{n+1} d(fx_1, fx_2) + (n+1)M^{n+2}\} + ...$$
$$+\{M^{n+p-1} d(fx_1, fx_2) + (n+p-1)M^{n+p}\}$$
$$\leq \sum_{i=n}^{n+p-1} M^i d(fx_1, fx_2) + \sum_{i=n}^{n+p-1} iM^{i+1}$$

It follows that the sequence $\{y_n\}$ is Cauchy sequence. Hence there exists z in X such that $y_n \to z$. Therefore $fx_{2k+1} \to z$ and $fx_{2k+2} \to z$. From (2), we have

$$f^2 x_{2k+1} = ffx_{2k+1} \in fSx_{2k} \subset Sfx_{2k},$$

And

$$f^2 x_{2k+2} = ffx_{2k+2} \in fTx_{2k+1} \subset Tfx_{2k+1}.$$

Now using lemma 2. 1

$$[d(fz, Sz)]^2 \leq [d(fz, f^2x_{2k+2}) + d(f^2x_{2k+2}, Sz)]^2$$
$$\leq [d(fz, f^2x_{2k+2}) + H(Tfx_{2k+1}, Sz)]^2$$
$$= [d(fz, f^2x_{2k+2})]^2 + 2H(Tfx_{2k+1}, Sz)d(fz, f^2x_{2k+2})$$
$$+[H(Tfx_{2k+1}, Sz)]^2$$
$$\leq [d(fz, f^2x_{2k+2})]^2 + 2H(Tfx_{2k+1}, Sz)d(fz, f^2x_{2k+2})$$
$$+ \alpha[d(fz, Sz)d(f^2x_{2k+1}, Tfx_{2k+1}) + d(fz, Tfx_{2k+1})$$
$$d(f^2x_{2k+1}, Sz)]$$
$$+ \beta[d(fz, Sz)d(f^2x_{2k+1}, Sz)$$
$$+ d(f^2x_{2k+1}, Tfx_{2k+1})d(fz, Tfx_{2k+1})]$$
$$+ \gamma[d(fz, Sz) + d(f^2x_{2k+1}, Tfx_{2k+1})]H(Tfx_{2k+1}, Sz)$$
$$\leq [d(fz, f^2x_{2k+2})]^2 + 2H(Tfx_{2k+1}, Sz)d(fz, f^2x_{2k+2})$$
$$+ \alpha[d(fz, Sz)d(f^2x_{2k+1}, f^2x_{2k+2}) + d(fz, f^2x_{2k+2})$$
$$d(f^2x_{2k+1}, Sz)]$$
$$+ \beta[d(fz, Sz)d(f^2x_{2k+1}, Sz)$$
$$+ d(f^2x_{2k+1}, f^2x_{2k+2})d(fz, f^2x_{2k+2})]$$
$$+ \gamma[d(fz, Sz) + d(f^2x_{2k+1}, f^2x_{2k+2})]d(f^2x_{2k+2}, Sz)$$

Since f is continuous, by letting $K \to \infty$, we obtain

$$[d(fz, Sz)]^2 \leq (\beta + \gamma)[d(fz, Sz)]^2$$

or

$$d(fz,Sz) \le (\sqrt{\beta+\gamma})d(fz,Sz).$$

Thus $fz \in Sz$. similarly,

$$[d(fz,Tz)]^2 \le [d(fz,f^2x_{2k+1}) + d(f^2x_{2k+1},Tz)]^2$$
$$\le [d(fz,f^2x_{2k+1}) + H(Sfx_{2k},Tz)]^2$$
$$\le \beta[d(fz,Sz)]^2$$

Therefore $fz \in Tz$. Hence *Z* is a coincidence point of *f* and *S* and *f* and *T*.

Corollary 2. 3: Let *S*, *T* be continuous mappings from a complete metric space *X* into *CB* (*X*) and $f \in C_S \cap C_T$ be a continuous mapping. Assume that (1) is satisfied. If $f(z) \in Sz \cap Tz$ implies $\lim_{n\to\infty} f^n z = t$, then *t* is a common fixed point of *S*, *T* and *f*.

Proof: Clearly, $fx \in Sz$ implies that $f^2z \in fSz \subset Sfz$. Therefore $f^{n+1}z \in Sf^nZ$. If follows that $t \in St$. Similarly $t \in Tt$. Moreover.

$$ft = f\lim_{n\to\infty} f^n z = \lim_{n\to\infty} f^{n+1}z = t.$$

Hence *t* is a common fixed point of *f*, *S* and *T*.

In the following theorem the continuity of *f* and its commutativity with *S* and *T* are not required.

Theorem 2. 4: Let *S*, *T* be two mappings from a metric space *X* into *CB* (*X*) and let *f* : $X \to X$ be a mapping such that *f* (*X*) is complete, $T(X) \subset f(X)$ and $S(X) \subset f(X)$. Suppose that (1) is satisfied, then there exists a common coincidence point of *f* and *T* and *f* and *S*.

Proof: As in the proof of theorem 2. 2 we construct the Cauchy sequence $y_n = fx_n \in X$. By our hypothesis it follows that there exists a point *u* in *X* such that $y_n \to z = fu$. Now using Lemma 2. 1, we have

$$[d(fu,Tu)]^2 \le [d(fu,fx_{2k+1}) + d(fx_{2k+1},Tu)]^2$$
$$\le [d(fu,fx_{2k+1}) + H(Sx_{2k},Tu)]^2$$
$$\le [d(fu,fx_{2k+1})]^2 + 2H(Sx_{2k},Tu)d(fu,fx_{2k+1})$$
$$+[H(Sx_{2k},Tu)]^2$$
$$\le [d(fu,fx_{2k+1})]^2 + 2d(fu,fx_{2k+1})H(Sx_{2k},Tu)$$
$$+\alpha[d(fx_{2k},Sx_{2k})d(fu,Tu) + d(fx_{2k},Tu)$$
$$d(fu,Sx_{2k})]$$

$$+\beta[d(fx_{2k},Sx_{2k})d(fu,Sx_{2k})+d(fu,Tu)$$
$$d(fx_{2k},Tu)]+\gamma[d(fx_{2k},Sx_{2k})+d(fu,Tu)]H(Sx_{2k},Tu)$$
$$\leq [d(fu,fx_{2k+1})]^2+2d(fu,fx_{2k+1})d(fx_{2k+1},Tu)$$
$$+\alpha[d(fx_{2k},fx_{2k+1})d(fu,Tu)+d(fx_{2k},Tu)$$
$$d(fu,fx_{2k+1})]$$
$$+\beta[d(fx_{2k},fx_{2k+1})d(fu,fx_{2k+1})+d(fu,Tu)$$
$$d(fx_{2k},Tu)]+\gamma[d(fx_{2k},fx_{2k+1})+d(fu,Tu)]d(fx_{2k+1},Tu).$$

Letting $k\to\infty$, we obtain

$$[d(fu,Tu)]^2\leq(\beta+\gamma)[d(fu,Tu)]^2$$

or $d(fu,Tu)\leq(\sqrt{\beta+\gamma})d(fu,Tu)$

Hence $fu\in Tu$. Similarly,

$$[d(fu,Su)]^2\leq[d(fu,fx_{2k+2})+d(fx_{2k+2},Su)]^2$$
$$\leq d(fu,fx_{2k+2})+H(Tx_{2k+1},Su)]^2$$
$$\leq(\beta+\gamma)[d(fu,Su)]^2$$

Hence $fu\in Su$.

Example: Let $S(x)=x^2$ and $T(x)=3-2x$ with $x=R$.

$$\left|S(x_n)-T(x_n)\right|=\left|x_n^2-3+2x_n\right|\to 0$$

if and only if $x_n\to 1$ and

$$\lim_n\left|ST(x_n)-TS(x_n)\right|=\lim_n 6\left|x_n-1\right|^2$$
$$=0 \text{ if } x_n\to 1$$

Thus *S* and *T* are compatible but not weakly commuting pair.

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