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Numerical analysis for some stochastic delay differential equations

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Abstract

In this paper we study the numerical solution of the stochastic delay differential equations of the following form

$$du(x, t) = f(x, t, u(x, t - \tau)\, dt + g(x, t, u(x, t), u(x, t - \tau)\, dW(t), \qquad t \in [0, T]$$

$$u(x, t) = \Psi(x, t), \qquad t \in [-\tau, 0]$$

(with the *lag* $\tau > o$) where $x \in R^v$, (R^v is the *v – dimensional* Euclidean space). Here $u \in R^n$, W(t) is an n-dimensional Wiener process given on the filtered probability space (Ω, F, P),

$f : R^{n+v+1} \to R^n$, $g : R^{n+v+1} \to R^{n\,x\,n}$ *and* $\Psi : R^v \times [-\tau, 0] \to R^n$.

Keywords: Stochastic delay differential equations, explicit one-step methods, conditional expectation, Lipschitz condition and linear growth condition.

AMS Subject Classifications: 65C30, 60H20, 60H10.

1. Introduction

Let (Ω, F, $\{F_t\}_{t\geq 0}$, P) be a complete probability space with the filtration $\{F_t\}_{t\geq 0}$ satifying the usual conditions (that is increasing and right continuous, and $\{F_t\}$, $t \geq 0$, contains all P-null sets in F). Thgoughout this paper let |.| denote the Euclidean vector norm and we shall use the notation.

$$\sup_x |u(x, t)| = \| u(., t) \| \text{ and } \sup_x |v(x, t)| = \| v(., t) \|.$$

The Lp – *norm* of a vector-valued L_p–integrable random variable $Z \in L_p(\Omega, F, P)$ will be denoted by

$$||Z||p = (E\ ||Z||p)1/p,\ 1 \le p < \infty,$$

where E is the expectation with respect to P.

The Numerical solutions of stochastic delay differential equations or SDDEs are studied in many papers (see [1-10]).

In this paper let W(t) be a n-dimensional Wiener process given on the filtered probability space (Ω, F, P). We consider the stochastic delay differential equation ($0=t_0 < T < \infty$):

$$du(x, t) = f(x, t, u(x, t), u(x, t-\tau))\, dt + g(x, t, u(x, t), u(x, t-\tau)dW(t), \qquad t \in [0, T]$$

$$u(x, t) = \Psi(x, t), \qquad t \in [-\tau, 0] \qquad (1.1)$$

with fixed lag, where $\Psi(x, t)$ is an F_{t0}-measurable $C(R^v \times [-\tau, 0], R^n]$ valued random variable such that

$$E\ ||\Psi|| < \infty.$$

($C(R^v$ x $[-\tau, 0], R^n)$) is the Banach space of all continuous paths from R^v x $[-\tau, 0] \to R^n$ equipped with the supremum norm

$$||\eta||\,C = \sup_{t \in [-\tau, 0]} ||\eta(.,t)||$$

where

$$||\eta(.,t)|| = \sup_x = |\eta(.,t)|.$$

If the functions f and g should not expilictly contain x and t the equaiton is called autonomous, and we consider this case for simplicity. Equation (1.1) can then be formulated equivalenty as

$$u(x, t) = u_0(x) + \int_0^t f(u(x, s), u(x, s-\tau)\, ds + \int_0^t g(u(x, s), u(x, s-\tau)dW(s), \qquad (1.2)$$

For $t \in [0, T]$ and with

$$u(x, t) = \Psi(x, t),\ for\ t \in [-\tau, 0].$$

We have $f : R^{n+v+1} \to R^n$, g : $R^{n+v+1} \to R^{nxn}$ and $\Psi : R^v \times [-\tau, 0] \to R^n$ and we need the following conditions:

(A1) The functions f and g are continous

(A2) The functions f and g satisfy a uniform Lischitz condition; that is there exists positive constants L_1, L_2, L_3 and L_4 such that ϕ_1, ϕ_2, Ψ_1, $\Psi_2 \in R^n$ and $t \in [0, T]$,

$$|| f(\phi_1, \Psi_1) - f(\phi_2, \Psi_2) || \le L_1 || \phi_1 - \phi_2 || + L_2 || \Psi_1 - \Psi_2 ||, \qquad (1.3)$$

and

$$||g(\phi_1, \Psi_1) - g(\phi_2, \Psi_2) || \le L_3 || \phi_1 - \phi_2 || + L_4 || \Psi_1 - \Psi_2 ||. \qquad (1.4)$$

(A3) the function Ψ is Holder continuous with exponent γ; that is, such that for t, s $\in [-\tau, 0]$

$$E \parallel \Psi(.,t) - \Psi(.,s) \parallel^{p} \leq L_5 \parallel t - s \parallel^{p\gamma},\ p = 1, 2 \qquad (1.5)$$

(A4) The functions f and g satify a linear growth conditon; that is, there exist positive constants K_1 and K_2 such that for all ϕ, ϕ_1, Ψ, $\Psi_1 \in R^n$ and $t \in [0, T]$,

$$\parallel f(\phi, \phi_1) \parallel^2 \leq K_1 (1 + \parallel \phi \parallel^2 + \parallel \phi_1 \parallel^2), \qquad (1.6)$$

$$\parallel g(\Psi, \Psi_1) \parallel^2 \leq K_2 (1 + \parallel \Psi \parallel^2 + \parallel \Psi_1 \parallel^2). \qquad (1.7)$$

2. Numerical analysis for an autonomous SDDEs

For simplicity we shall in the sequel consider equation (1.1) in the authonomous form: that is, we shall work with

$$du(x,t) = f(x, t, u(x,t), u(x, t-\tau)\, dt + g(x, t, u(x,t), u(x, t-\tau)\, dW(t), \quad t \in [0, T]$$

$$u(x,t) = \Psi(x,t), \qquad t \in [-\tau, 0]. \qquad (2.1)$$

We define a mesh with a uniform step on the interval [0, T], h = T/N, $t_n = n.h$, where $n = 0, \ldots, N$, and where we assume that for the given h there is a corresponding integer N_τ such that the lag can be expressed in terms of the step size as $\tau = N_\tau.h$. We consider strong approximations $v_n(x)$ of the solution to equation (2.1), using a stochastic explicit one-step method of the form

$$v_{n+1}(x) = v_n(x) + \phi(h, v_n(x), v_{n} - N_\tau(x), I_\phi),\ n = 0, \ldots, N-1, \qquad (2.2)$$

where the initial values are given by

$$v_{n-N\tau}(x) = \Psi(x, t_n - \tau)\ for\ n - \tau \leq 0.$$

The increment function $\phi(h, ., ., I_\phi) : R^n \times R^n \to R^n$ incorporates a finit number of multiple Ito-integrals (see [11]) of the form

$$I_{(j1, \ldots, jl), h} = \int_t^{t+h} \int_t^{s1} \ldots \int_t^{s2} dW^{j1}(s_1) \ldots dW^{ji-1}(s_{l-1}) dw^{jl}(s_l),$$

Where $j_i \in$ 0, 1 and $dW^0(t) = dt$ with $t = t_n$ in the case (2.1). We denote by I_ϕ the collection of Ito-integrals required to compute the increment function ϕ. The increment function ϕ is assumed to generate approximations $v_n(x)$ which are F_{tn} – measurable. We suppose there exist positive constants C_1, C_2, C_3 such that ζ, ζ^1 $\eta - \eta^1 \in R^n$,

$$\parallel E(\phi(h, \zeta, \eta, I_\phi) - \phi(h, \zeta^1, \eta^1, I_\phi)) \parallel \leq C_1 h\, (\parallel \zeta - \zeta^1 \parallel + \parallel \eta - \eta^1 \parallel, \qquad (2.3)$$

$$E(\parallel \phi(h, \zeta, \eta, I_\phi) - \phi(h, \zeta^1, \eta^1, I\phi) \parallel^2) \leq C_2 h\, (\parallel \zeta - \zeta \parallel^2 + \parallel \eta - \eta^1 \parallel^2). \qquad (2.4)$$

and

$$E(\parallel \phi(h, \zeta, \eta, I_\phi) \parallel^2) \leq C_3 h\, (1 + \parallel \zeta \parallel^2 + \parallel \eta \parallel^2). \qquad (2.5)$$

Notation 2.1. We denote by $u(x, t_{n+1})$ the value of the exact solution of equation (2.1) at the meshpoint t_{n+1}, by $v_{n+1}(x)$ the value of the approximate solution using equation (2.2), and by $v(x, t_{n+1})$ the value obtained after just one step of equation (2.2): that is,

$$v(x, t_{n+1}) = v(x, t_n) + \phi(h, v(x, t_n), u(x, t_n - \tau), I_\phi).$$

Definition 2.1. The error of the above approximation $v_n(x)$ on the meshpoints is the sequence of random variables

$$\in_n := v(x, t_n) - v_n(x), \; n = 1, \ldots, N. \tag{2.6}$$

Note that $\in_n$ is F_{tn}-measurable since both $u(x, t_n)$ and $v_n(x)$ are F_{tn}-measurable random variables, and that $(E \| \in_n \|2)^{½}$ is the L^2-norm of (2.6).

Definition 2.2. Let

$$\delta_{n+1}(x) = u(x, t_{n+1}) - v(x, t_{n+1}), \qquad n = 0, \ldots, N-1. \tag{2.7}$$

The method (2.2) is said to be consistent with order p_1 in the mean and with order p_2 in the mean-square sense if, with

$$P_2 \geq \frac{1}{2} \text{ and } p_1 \geq p_2 + \frac{1}{2}, \tag{2.8}$$

the estimates

$$\max_{0 \leq n \leq N-1} \| E \| \delta_{n+1}(.) \| ch^{p2} \text{ as } h \to 0, \tag{2.9}$$

and

$$\max_{0 \leq n \leq N-1} (E \| \delta_{n+1}(.) \|^2)^{½} \leq ch^{p2} \text{ as } h \to 0, \tag{2.10}$$

hold, where the (generic) constant C does not depend on h, but may depend on T, and on the initial data. We now state the main theorem of this paper

Theorem 2.1. We assume that the functions f and g satisfy the conditions from A1 to A4. Suppose that the method defined by equation (2.2) is consistent with order p_1 in the mean and order p_2 in the mean-square sense, with p_1 and p_2 satisfying inequality (2.8), and the increment function ϕ in equation (2.2) satisfies the estimates (2.3) and (2.4). Then the approximation (2.2) for equation (2.1) is convergent in L^2 (as $h \to 0$ with $\tau/h \in N$) with order $p = p_1 - \frac{1}{2}$. That is, convergence is in the mean-square sense, and

$$\max_{1 \leq n \leq N} (E \| \in_n \|^2)^{½} \leq ch^p \text{ as } h \to 0. \tag{2.11}$$

Proof. Using Notation 2.1, adding and subtracting $u(x, t_n)$ and $\phi(h, u(x, t_n), u(x, tn - \tau), I\phi)$, and rearranging, we obtain

$$\epsilon_{n+1} = u(x, t_{n+1}) - v_{n+1}(x)$$

$$= u(x, t_n) - v_n(x) + u(x, t_{n+1}) - u(x, t_n) - \phi(h, u(x, t_n), u(x, t_n - \tau), I_\phi)$$

$$+ \phi(h, u(x, t_n), u(x, t_n - \tau), I_\phi) - \phi(h, v_n(x), v_{n-N_\tau}(x), I_\phi)$$

$$= \epsilon_n + \delta_{n+1} + z_n,$$

where

$$\epsilon_n = u(x, t_n) - v_n(x),$$

$$\delta_{n+1} = u(x, t_{n+1}) - u(x, t_n) - \phi(h, u(x, t_n), u(x, t_n - \tau), I_\phi)$$

and

$$z_n = \phi(h, u(x, t_n), u(x, t_n - \tau), I_\phi) - \phi(h, v_n(x), v_{n-N_\tau}(x), I_\phi). \tag{2.12}$$

Thus, squaring, employing the conditional means with respect to the σ-algebra F_{t0} and taking the norm, we obtain

$$E(\| \epsilon_{n+1} \|^2 |F_{t_0}) \leq E(\| \epsilon_n \|^2 |F_{t_0}) + E(\| \delta_{n+1} \|^2 |F_{t_0}) + E(\| z_n \|^2 |F_{t_0})$$

$$+2 \| E(\delta_{n+1}.\epsilon_n|F_{t_0}) \| +2 \| E(\delta_{n+1}.z_n|F_{t_0}) \| +2 \| E(\epsilon_n.z_n|F_{t_0}) \|, \tag{2.13}$$

which holds almost surely.

We shall now estimate the separate terms in inequality (2.13) individually and in sequence; all the estimates hold almost surely. We shall frequently use the Holder inequality, the inequality $2ab \leq a^2 + b^2$ and properties of conditional expectation, which can be found in [12]. In the sequel we shall use c to denote an unspecified constant, which depends only on the constants L_1, L_2, L_3, L_4, k_1, k_2, c_1 *and* c_2, and on T and the initial data.

Due to the assumed consistency in the mean-square sence of the method, we have

$$E(\| \delta_{n+1} \|^2 |F_{t_0}) = E(E(\| \delta_{n+1} \|^2 |F_{t_n})|F_{t_0}) \leq ch^{2p_2}.$$

Due to property (2.4) of the increment function, we have

$$E(\| z_n \|^2 |F_{t_0}) \leq chE(\| \epsilon_n \|^2 |F_{t_0}) + chE(\| \epsilon_{n-N_\tau} \|^2 |F_{t_0}).$$

We have, due to the consistency condition,

$$2 \| E(\delta_{n+1}.\epsilon_n|F_{t_0} \| \leq 2 \| E(E(\delta_{n+1}|F_{t_n})\epsilon_n|F_{t_0}) \| \leq 2(E \| E(\delta_{n+1}|F_{t_n}) \|^2)^{\frac{1}{2}}.(E(\| \epsilon_n \|^2 |F_{t_0}))^{\frac{1}{2}}$$

$$= 2(E(ch^{2p_1-1}))^{\frac{1}{2}}.(hE(\| \epsilon_n \|^2 |F_{t_0}))^{\frac{1}{2}} \leq ch^{2p_1-1} + hE(\| \epsilon_n \|^2 |F_{t_0}).$$

By employing the consistency condition and property (2.4) of the increment function ϕ, we have

$$2 \| E(\delta_{n+1}.z_n|F_{t_0}) \| \leq 2(E(\| \delta_{n+1} \|^2 |F_{t_0}))^{\frac{1}{2}}(E(\| z_n \|^2 |F_{t_0}))^{\frac{1}{2}}$$

$$\leq E(E(\| \delta_{n+1} \|^2 |F_{t_n})|F_{t_0}) + E(\| z_n \|^2 |F_{t_0}) \leq ch^{2p_2} + chE(\epsilon_n^2|F_{t_0}) + chE(\epsilon_{n-N_\tau}^2|F_{t_0}).$$

Using definition (2.12) and property (2.3) of the increment function ϕ, we have

$$2 \parallel (z_n.\epsilon_n|F_{t_0}) \parallel \leq 2E(\parallel E(z_n|F_{t_n}) \parallel . \parallel \epsilon_n \parallel |F_{t_0}) \leq chE(\parallel \epsilon_n \parallel^2 |F_{t_0}) + 2chE(\parallel \epsilon_n \parallel \parallel \epsilon_{n-N_\tau} \parallel |F_{t_0})$$

$$\leq chE(\parallel \epsilon_n \parallel^2 |F_{t_0}) + ch[2(E(\parallel \epsilon_n \parallel^2 |F_{t_0}))^{\frac{1}{2}}.(E(\parallel \epsilon_{n-N_\tau} \parallel^2 |F_{t_0}))^{\frac{1}{2}}]$$

$$\leq chE(\parallel \epsilon_n \parallel^2 |F_{t_0}) + chE(\parallel \epsilon_n \parallel^2 |F_{t_0}) + chE(\parallel \epsilon_{n-N_\tau} \parallel^2 |F_{t_0})$$

$$\leq chE(\parallel \epsilon_n \parallel^2 |F_{t_0}) + chE(\parallel \epsilon_{n-N_\tau} \parallel^2 |F_{t_0}).$$

Combining these results with $2_{p2} \leq 2_{p1} - 1$, we obtain

$$E(\epsilon_{n+1}^2|F_{t_0}) \leq (1+ch)E(\epsilon_n^2|F_{t_0}) + ch^{2p_2} + chE(\parallel \epsilon_{n-N_\tau} \parallel^2 |F_{t_0}).$$

Now we shall prove the assertion by an induction argument over consecutive intervals of length τ up to the end ot the interval $[0, T]$. Since we have exact initial values, we set

$$\epsilon_n = 0 \ for \ n = -N_\tau, ..., 0.$$

Step 1. Suppose that $t_n \in [0, \tau]$; that is, n = 1, …, $N\tau$ and $\in_{n-N\tau} = 0$.

$$E(\epsilon_{n+1}^2|F_{t_0}) \leq (1+ch)E(\epsilon_n^2|F_{t_0}) + ch^{2p_2} \leq ch^{2p_2}\sum_{k=0}^{n}(1+ch)^k$$

$$= ch^{2p_2}\frac{(1+ch)^{n+1}-1}{(1+ch)-1} \leq ch^{2p_2-1}((e^{ch})^{n+1}-1) \leq ch^{2p_2-1}(e^{cT}-1).$$

Step 2. Suppose that $t_n \in [k\tau, (k+1)\tau]$, and make the asumption that

$E(\parallel \in_{n-N\tau} \parallel^2 | F_{t0}) \leq ch^{2p2-1}$.

Then

$$E(\epsilon_{n+1}^2|F_{t_0}) \leq (1+ch)E(\epsilon_n^2|F_{t_0}) + ch^{2p_2} + chE(\parallel \epsilon_{n-N_\tau} \parallel^2 |F_{t_0}) \leq (1+ch)E(\epsilon_n^2|F_{t_0}) + ch^{2P_2} + hch^{2p_2-1}$$

$$= (1+ch)E(\epsilon_n^2|F_{t_0}) + ch^{2p_2} \leq ch^{2p_2-1}(e^{cT}-1),$$

by the same arguments as above. This, implies, almost surely, that

$$(E(\epsilon_{n+1}^2|F_{t_0}))^{\frac{1}{2}} \leq ch^{2p_2-\frac{1}{2}},$$

which proves the theorem

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