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A unifed generalization of perturbed trapezoid and mid-point inequalities and applications in numerical integration

Wen-bing Chen, Qun Chen and Wen-jun Liu*

College of Mathematics and Physics,
Nanjing University of Information Science and Technology,
Nanjing 210044, China
**Corresponding author: E-mail: wjliu@nuist.edu.cn*

Abstract

A unifed generalization of perturbed trapezoid and mid-point in-equalities is obtained. Applications in numerical integration are also given.

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1. Introduction

In recent years a number of authors [1, 2, 4, 5, 7, 8, 9] have considered the following perturbed trapezoid and mid-point inequalities

$$\left|\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \\ \leq C(\Gamma-\gamma)(b-a)^3, \tag{1}$$

$$\left|\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)(b-a) - \frac{(b-a)^2}{24}[f'(b)-f'(a)]\right| \\ \leq C(\Gamma-\gamma)(b-a)^3, \tag{2}$$

where $f : [a, b] \to R$ is a twice differentiable function and there exist constants $\gamma, \Gamma \in R$, with $\gamma \leq f''(t) \leq \Gamma$, $t \in [a, b]$ while C is a constant. We can find

$C = \frac{1}{8}$, $C = \frac{1}{6\sqrt{5}}$, $C = \frac{1}{18\sqrt{3}}$, $C = \frac{1}{32}$ and $C = \frac{1}{24\sqrt{5}}$, respectively.

Recently, Cheng and Sun [3], Liu [6] showed that the best constants in (1) and (2) are both $C = \frac{1}{36\sqrt{3}}$ by defining

$$p(t) = -\frac{1}{2}(t-a)(b-t)+\frac{(b-a)^2}{12} \text{ and } q(t) = \begin{cases} \frac{1}{2}(t-a)^2 - \frac{(b-a)^2}{24}, & t \in [a, \frac{a+b}{2}], \\ \frac{1}{2}(t-b)^2 - \frac{(b-a)^2}{24}, & t \in (\frac{a+b}{2}, b], \end{cases}$$

respectively.

In this paper, we obtain a generalization of perturbed trapezoid and mid-point inequalities by defning a uniform r(t) as in (3). Our result in special case yield the above existing results. Finally, we give applications in numerical integration.

2. Main Results

Theorem 1. *Let $I \subset R$ be an open interval, $a, b \in I$, $a < b$. If $f : I \to R$ is a twice differentiable function such that f'' is integrable and there exist constants γ, $\Gamma \in R$; with $\gamma \le f''(t) \le \Gamma$, $t \in [a, b]$, $0 \le \lambda \le 1$. Then we have*

$$\left| \int_a^b f(t)dt - (b-a)\left[(1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2}\right] - \frac{1-3\lambda}{24}(b-a)^2[f'(b)-f'(a)] \right|$$

$$\le \begin{cases} \frac{2\left(\lambda^2-\lambda+\frac{1}{3}\right)^{\frac{3}{2}} + \lambda(1-\lambda)(1-2\lambda)}{24}(\Gamma-\gamma)(b-a)^3, & \lambda \in \left[0, \frac{1}{3}\right], \\ \frac{\left(\lambda^2-\lambda+\frac{1}{3}\right)^{\frac{3}{2}}}{6}(\Gamma-\gamma)(b-a)^3, & \lambda \in \left(\frac{1}{3}, \frac{2}{3}\right], \\ \frac{2\left(\lambda^2-\lambda+\frac{1}{3}\right)^{\frac{3}{2}} + \lambda(1-\lambda)(2\lambda-1)}{24}(\Gamma-\gamma)(b-a)^3, & \lambda \in \left(\frac{2}{3}, 1\right]. \end{cases}$$

Proof. Let $r : [a, b] \to R$ be given by

$$r(t) = \begin{cases} \frac{1}{2}(t-a)[t-(1-\lambda)a-\lambda b] - \frac{1-3\lambda}{24}(b-a)^2, & t \in [a, \frac{a+b}{2}], \\ \frac{1}{2}(b-t)[\lambda a+(1-\lambda)b-t] - \frac{1-3\lambda}{24}(b-a)^2, & t \in (\frac{a+b}{2}, b]. \end{cases} \tag{3}$$

Integrating by parts, we have

$$\int_a^b r(t)f''(t)dt = \int_a^b f(t)dt - (b-a)\left[(1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2}\right] - \frac{1-3\lambda}{24}(b-a)^2[f'(b)-f'(a)] \quad (4)$$

and

$$\int_a^b r(t)dt = 0. \quad (5)$$

If C is a constant then from (4) and (5) it follows

$$\int_a^b r(t)[f''(t)-C]dt = \int_a^b f(t)dt - (b-a)\left[(1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2}\right] - \frac{1-3\lambda}{24}(b-a)^2[f'(b)-f'(a)]. \quad (6)$$

If we choose $C = \frac{\gamma+\Gamma}{2}$, then we have

$$\left|\int_a^b f(t)dt - (b-a)\left[(1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2}\right] - \frac{1-3\lambda}{24}(b-a)^2[f'(b)-f'(a)]\right| \le \frac{\Gamma-\gamma}{2}\int_a^b |r(t)|dt, \quad (7)$$

and it is easy to calculate by substitution $t = a + \frac{b-a}{2}(\lambda+t)$ that

$$\int_a^b |r(t)|dt = 2\int_a^{\frac{a+b}{2}} \left|\frac{1}{2}\left[(x-a) - \frac{\lambda}{2}(b-a)\right]^2 - \frac{3\lambda^2-3\lambda+1}{24}(b-a)^2\right| dx = \frac{(b-a)^3}{24}\int_{-\lambda}^{1-\lambda} \left|3t^2 - (3\lambda^2-3\lambda+1)\right| dt \quad (8)$$

From (7) and (8) we get the result.

Remark 2. *We note that in the special cases, if we take $\lambda = 1$ and $\lambda = 0$ in Theorem 1 respectively, we get Theorem 1.2 in [3] and Theorem 1 in [6].*

Corollary 3. Under the assumptions of Theorem 1 and with $\lambda = \frac{1}{2}$, we have the inequality

$$\left|\int_a^b f(t)dt - \frac{1}{2}f\left(\frac{a+b}{2}\right)(b-a) - \frac{1}{2}\frac{f(a)+f(b)}{2}(b-a) + \frac{1}{48}(b-a)^2[f'(b)-f'(a)]\right| \le \frac{1}{144\sqrt{3}}(\Gamma-\gamma)(b-a)^3. \tag{9}$$

Corollary 4. *Under the assumptions of Theorem 1 and with $\lambda = \frac{1}{3}$, we have the Simpson inequality*

$$\left|\int_a^b f(t)dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \le \frac{1}{162}(\Gamma-\gamma)(b-a)^3. \tag{10}$$

Corollary 5. Under the assumptions of Theorem 1 and with $\lambda = \frac{2}{3}$, we have the inequality

$$\left|\int_a^b f(t)dt - \frac{1}{3}f\left(\frac{a+b}{2}\right)(b-a) - \frac{2}{3}\frac{f(a)+f(b)}{2}(b-a) + \frac{1}{24}(b-a)^2[f'(b)-f'(a)]\right| \le \frac{1}{162}(\Gamma-\gamma)(b-a)^3. \tag{11}$$

3. Applications in numerical integration

Theorem 6. *Let the assumptions of Theorem 1 hold. If $D = \{a = x_0 < x_1 < \ldots < x_n = b\}$ is a given division of the interval $[a; b]$ then we have*

$$\int_a^b f(t)dt = A_{MT}(f,D) + R_{MT}(f,D),$$

where

$$A_{MT}(f,D) = \sum_{i=0}^{n-1} h_i\left[(1-\lambda)f\left(\frac{x_i+x_{i+1}}{2}\right) + \lambda\frac{f(x_i)+f(x_{i+1})}{2}\right] + \frac{1-3\lambda}{24}\sum_{i=0}^{n-1} h_i^3[f'(x_{i+1}) - f'(x_i)]$$

$$|R_{MT}(f,D)| \le \begin{cases} \dfrac{2\left(\lambda^2-\lambda+\frac{1}{3}\right)^{\frac{3}{2}}+\lambda(1-\lambda)(1-2\lambda)}{24}(\Gamma-\gamma)\displaystyle\sum_{i=0}^{n-1}h_i^3, & \lambda\in\left[0,\frac{1}{3}\right], \\ \dfrac{\left(\lambda^2-\lambda+\frac{1}{3}\right)^{\frac{3}{2}}}{6}(\Gamma-\gamma)\displaystyle\sum_{i=0}^{n-1}h_i^3, & \lambda\in\left(\frac{1}{3},\frac{2}{3}\right], \\ \dfrac{2\left(\lambda^2-\lambda+\frac{1}{3}\right)^{\frac{3}{2}}+\lambda(1-\lambda)(2\lambda-1)}{24}(\Gamma-\gamma)\displaystyle\sum_{i=0}^{n-1}h_i^3, & \lambda\in\left(\frac{2}{3},1\right], \end{cases}$$

and $h_i = x_{i+1} - x_i$, $i = 0, 1, 2, \ldots, n-1$.

Proof. Apply Theorem 1 to the interval $[x_i, x_{i+1}]$, $i = 0, 1, 2, \ldots, n-1$ and sum. Then use the triangle inequality to obtain the desired result.

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