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Operator Equations and Regularization of Approximation Problems in Reproducing Kernel Hilbert Spaces

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Abstract

In [5], the authors considered a system of bounded linear operators $L_j : H \to H_j$, $(j = 1, 2, \ldots)$ of Hilbert spaces, where H is a Hilbert space consisting of complex valued functions $\{f\}$ on a set E. To discuss about the best solution of the approximation problem: for given $g_j \in H_j$

$$\inf_{f \in H} \sum_j \|L_j f - g_j\|_{H_j}^2 \tag{1}$$

they assumed a reproducing kernel Hilbert space structure in H with inner product defined by $\sum_j \langle L_j f_1, L_j f_2 \rangle_{H_j}$. If $\sum_j \langle L_j f_1, L_j f_2 \rangle_{H_j}$ does not define an inner product for a reproducing kernel Hilbert space they propose to add one or more terms considering known reproducing Kernel Hilbert space. In this paper, we replace H by a reproducing kernel Hilbert space H_K and discuss approximate solutions of the problem (1) in a natural way. We shall also give a method for approximate solutions of the problem (1) and give a concrete example with graph.

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1. Introduction

Let H_K be a reproducing kernel Hilbert space comprising complex-valued functions $\{f\}$ on E and let L_j be a bounded linear operator on H_K into a Hilbert space H_j. Then, for $g_j \in H_j$ we shall consider the best approximation problem:

$$\inf_{f\in H_K} \sum_j \|L_j f - g_j\|^2_{H_j}. \tag{1.1}$$

The problem (1.1) is called **solvable** if there exists a function $\tilde{f} \in H_K$ such that

$$\inf_{f\in H_K} \sum_j \|L_j f - g_j\|^2_{H_j} = \sum_j \|L_j \tilde{f} - g_j\|^2_{H_j}$$

and $\tilde{f}$ is called a **solution** of (1.1). Among the solutions one with minimum norm in H_K is called a **best solution** of (1.1). In our situation best solution is unique. The best solution of (1.1) is sometimes called the **best approximate solution** of the system of operator equations

$$L_j f = g_j, \quad j = 1, 2, \ldots.$$

Let H be the underlying vector space of the Hilbert space H_K. In [5], the authors proved that if there is a reproducing kernel Hilbert space structure H_{K_L} in H defined by

$$\|f\|^2_{H_{K_L}} = \sum_j \|L_j f\|^2_{H_j}$$

then the problem (1.1) is always solvable for any $g_j \in H_j$. Thus, we see that a sufficient condition of solvability of (1.1) is $\left(\sum_j \|L_j f\|^2_{H_j}\right)^{\frac{1}{2}}$ defines a norm for a reproducing kernel Hilbert space in H. Given $g_j \in H_j$, whether the problem (1.1) is solvable or not we shall prove that the problem

$$\inf_{f\in H_K} \left(\lambda \|f\|^2_{H_K} + \sum_j \|L_j f - g_j\|^2_{H_j}\right) \tag{1.2}$$

is always solvable for any $\lambda > 0$. The problem (1.2) is called a **regularization** of the problem (1.1). For very small λ, we can think of the best solution of (1.2) as an approximation of the best solution of (1.1) though the best solution of (1.1) may not exist.

2. Background Theorems

Theorem 2.1. Let H_K be the reproducing kernel Hilbert space admitting the reproducing kernel $K(x, y)$ on a set E, $\mathcal{H}$ be any Hilbert space and $L : H_K \to \mathcal{H}$ be a bounded linear map. Let H_k be the Hilbert space admitting the reproducing kernel

$$k(x, y) = \big(L^*LK(., y), L^*LK(., x)\big)_{H_K}, \quad x, y \in E.$$

Then, for a member $g \in \mathcal{H}$, there exists a function $\tilde{f} \in H_K$ such that

$$\inf_{f \in H_K} \|Lf - g\|_{\mathcal{H}} = \|L\tilde{f} - g\|_{\mathcal{H}}$$

if and only if $L^*g \in H_k$. Furthermore, if the existence of an extremal function is ensured, then there exists a unique extremal function $\breve{f}$ with the minimum norm in H_K, and the function $\breve{f}$ is expressible in the form

$$\breve{f}(x) = \big(L^*g, L^*LK(., x)\big)_{H_k}, \quad x \in E.$$

Theorem 2.2. Let H_K be a Hilbert space admitting the reproducing kernel $K(x, y)$ on a set E and let H be its underlying vector space. Let $L : H_K \to \mathcal{H}$ be a bounded linear operator of Hilbert spaces. For $\lambda > 0$ introduce a structure in H and call it $H_{K\lambda}$ as

$$\langle f_1, f_2\rangle_{H_{K\lambda}} = \lambda\langle f_1, f_2\rangle_{H_K} + \langle Lf_1, Lf_2\rangle_{\mathcal{H}} \tag{2.1}$$

then $H_{K\lambda}$ is a reproducing kernel Hilbert space with reproducing kernel $K\lambda(x, y)$ on E satisfying the equation

$$K(., y) = (\lambda I + L^*L)K\lambda(., y), \tag{2.2}$$

where L^* is the adjoint of $L : H_K \to \mathcal{H}$.

Proof. It is clear that $\langle ., .\rangle_{H\lambda}$ defines an inner product in $H_{K\lambda}$ and every Cauchy sequence in $H_{K\lambda}$ is a Cauchy sequence in H_K. Moreover, using continuity of L it is easy to prove that every convergent sequence in H_K is a convergent sequence in $H_{K\lambda}$. Hence, $H_{K\lambda}$ is a Hilbert space. Since H_K is a reproducing kernel Hilbert space therefore for any fixed $x \in E$ there exists a positive number M_x such that

$$|f(x)| \leq M_x\|f\|_{H_K}$$

for every $f \in H_K$. Hence,

$$|f(x)|^2 \leq \frac{{M_x}^2}{\lambda}\lambda\|f\|^2_{H_K} \leq \frac{{M_x}^2}{\lambda}\|f\|^2_{H_{K\lambda}}$$

for every $f \in H_{K\lambda}$. Thus every point evaluation is bounded in $H_{K\lambda}$. Hence, $H_{K\lambda}$ is a reproducing kernel Hilbert space. Let $K\lambda(x, y)$ be the reproducing kernel of $H_{K\lambda}$. Then for any $f \in H_{K\lambda}$ and for any $y \in E$,

$$\begin{aligned}
& f(y) = \langle f, K\lambda(., y)\rangle_{H_{K\lambda}} \\
\Longleftrightarrow \quad & \langle f, K(., y)\rangle_{H_K} = \lambda\langle f, K\lambda(., y)\rangle_{H_K} + \langle Lf, LK\lambda(., y)\rangle_{\mathcal{H}} \\
\Longleftrightarrow \quad & \langle f, K(., y)\rangle_{H_K} = \lambda\langle f, K\lambda(., y)\rangle_{H_K} + \langle f, L^*LK\lambda(., y)\rangle_{H_K} \\
\Longleftrightarrow \quad & \langle f, K(., y)\rangle_{H_K} = \langle f, \lambda K\lambda(., y) + L^*LK\lambda(., y)\rangle_{H_K}.
\end{aligned}$$

Hence we have the theorem. ■

Corollay 2.3. Let H_K and H be as in Theorem 2.2 and let $L_j : H_K \to H_j$ be bounded linear operators of Hilbert spaces for $j = 1, 2, \ldots$, n. For $\lambda > 0$, introduce a structure in H and call it $H_{K\lambda}$ as

$$\langle f_1, f_2 \rangle_{H_{K\lambda}} = \lambda \langle f_1, f_2 \rangle_{H_K} + \sum_{j=1}^{n} \langle L_j f_1, L_j f_2 \rangle_{H_j} \tag{2.3}$$

then $H_{K\lambda}$ is a reproducing kernel Hilbert space with reproducing kernel $K\lambda(x, y)$ on E satisfying the equation

$$K(., y) = \left(\lambda I + \sum_{j=1}^{n} L_j^* L_j \right) K\lambda(., y), \tag{2.4}$$

where L_j^* is the adjoint of $L_j : H_K \to H_j$.

Proof. Define the dirrect sum space $\mathcal{H} = \oplus_{j=1}^{n} H_j$ and define $L : H_K \to \mathcal{H}$ by

$$Lf = (L_1 f, L_2 f, \ldots, L_n f), \quad f \in H_K.$$

Then L is a bounded linear operator with adjoint $L^* : \mathcal{H} \to H_K$ defined by $L^* g = \sum_{j=1}^{n} L_j^* g_j$, where $\mathbf{g} = (g_1, g_2, \ldots, g_n) \in \mathcal{H}$. Thus, we have $L^* L f = \sum_{j=1}^{n} L_j^* L_j f$. Hence the corollary follows from Theorem 2.2. ■

Theorem 2.4. Let H_K be a Hilbert space admitting reproducing kernel $K(x, y)$ on a set E. Let for $j = 1, 2, \ldots, m$, $L_j : H_K \to H_j$ be bounded linear operators of Hilbert spaces. Let H be the vector space underlying H_K. Suppose that there can be given a reproducing kernel Hilbert space structure in H and call it H_{K_L} by the definition

$$\langle f_1, f_2 \rangle_{H_{K_L}} = \sum_{j=1}^{m} \langle L_j f_1, L_j f_2 \rangle_{H_j}. \tag{2.5}$$

Then the best approximation problem:

$$\inf_{f \in H_K} \sum_{j=1}^{m} \| L_j f - g_j \|_{H_j}^2 \tag{2.6}$$

is always solvable for any $g_j \in H_j$; and its best solution is given by

$$f^*(x) = \sum_{j=1}^{m} \langle g_j, L_j K_L(., x) \rangle_{H_j}, \tag{2.7}$$

where $K_L(x, y)$ is the reproducing kernel of H_{K_L}.

Proof. Let $\mathcal{H} = \oplus_{j=1}^m H_j$ be the direct sum Hilbert space of H_j's. Clearly, $L_j : H_{K_L} \to H_j$ are bounded linear operators. Let L_j^* be the adjoint of $L_j : H_{K_L} \to H_j$. Define the linear operator $L : H_{K_L} \to \mathcal{H}$ by

$$Lf = (L_1 f, L_2 f, \cdots, L_m f).$$

Then, for $\mathbf{g} \in \mathcal{H}$, we have

$$L^*(\mathbf{g}) = \sum_{j=1}^m L_j^* g_j,$$

where $\mathbf{g} = (g_1, g_2, \cdots, g_m)$. Hence, by Theorem 2.1, the problem

$$\inf_{f \in H_{K_L}} \|Lf - \mathbf{g}\| \tag{2.8}$$

is solvable if and only if

$$\begin{aligned} (L^*\mathbf{g})(x) &= \langle L^*\mathbf{g}, K_L(\cdot, x)\rangle_{H_{K_L}} \\ &= \langle \mathbf{g}, LK_L(\cdot, x)\rangle_{\mathcal{H}} \\ &\in H_k \end{aligned} \tag{2.9}$$

where H_k is the Hilbert space admitting the reproducing kernel

$$k(x, y) = \langle L^*LK_L(\cdot, y), L^*LK_L(\cdot, x)\rangle_{H_{K_L}}. \tag{2.10}$$

From (2.4), we see that L is an isometry from H_{K_L} into $\mathcal{H}$ and so L^*L is an identity operator on H_{K_L}. Therefore, we have from (2.9)

$$k(x, y) = K_L(x, y).$$

Hence, we see that the condition (2.8) is always satisfied; and since $N(L^*L) = \{0\}$, the approximation problem (2.7) is uniquely solvable. By Theorem 2.1, the unique extremal function f^* is represented by

$$\begin{aligned} f^*(x) &= \langle L^*\mathbf{g}, K_L(\cdot, x)\rangle_{H_{K_L}} \\ &= \langle \mathbf{g}, LK_L(\cdot, x)\rangle_{\mathcal{H}} \\ &= \sum_{j=1}^m \langle g_j, L_j K_L(\cdot, x)\rangle_{H_j}.(\mathbf{??}) \end{aligned} \tag{2.11}$$

Since the approximation problems (2.5) and (2.7) are identical, the proof of the theorem is thus complete. ■

3. Main Theorem

Theorem 3.1. Let H_K be the reproducing kernel Hilbert space admitting the reproducing kernel $K(x, y)$ on a set E and let for $j = 1, 2, \ldots, n$, $L_j : H_K \to H_j$ be bounded linear operators of Hilbert spaces. Then, for any $\lambda > 0$ and for any $g_j \in H_j$, $j = 1, 2, \ldots, n$, the approximation problem

$$\inf_{f \in H_K} \left(\lambda \|f\|^2_{H_K} + \sum_j \|L_j f - g_j\|^2_{H_j} \right)$$

is solvable and its best solution $f^*\lambda$ is given by

$$f^*\lambda(x) = \sum_{j=1}^{n} \langle g_j, L_j K\lambda(., x)\rangle_{H_j}, \tag{3.1}$$

where $K\lambda(x, y)$ is the reproducing kernel of the Hilbert space $H_{K\lambda}$ in Corollary 2.3.

Proof. Setting $m = n + 1$, $H_m = H_K$, $L_m = \sqrt{\lambda} I$ and $0 = g_m \in H_K$ in Theorem 2.4, we have Theorem 3.1. ■

If $n = 1$, then we have the following corollary:

Corollary 3.2. Let H_K be the reproducing kernel Hilbert space admitting the reproducing kernel $K(x, y)$ on a set E and let $L : H_K \to \mathcal{H}$ be a bounded linear operator of Hilbert spaces. Then, for any $\lambda > 0$ and for any $g \in \mathcal{H}$, the approximation problem

$$\inf_{f \in H_K} \left(\lambda \|f\|^2_{H_K} + \|Lf - g\|^2_{\mathcal{H}} \right)$$

is solvable and its best solution $f^*\lambda$ is given by

$$f^*\lambda(x) = \langle g, LK\lambda(., x)\rangle_{\mathcal{H}}, \tag{3.2}$$

where $K\lambda(x, y)$ is the reproducing kernel of the Hilbert space $H_{K\lambda}$ in Theorem 2.2.

4. A Concrete Example

We consider the reproducing kernel Hilbert space H_K consisting of all complex valued functions on $\mathbb{R}$ equipped with the norm defined by

$$\|f\|^2_K = \int_{-\infty}^{\infty} \{|f(x)|^2 + |f'(x)|^2\} dx.$$

Then the reproducing kernel of H_K is given by

$$K(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\xi(x-y)}}{1 + \xi^2} d\xi.$$

It is clear that H_K is a subset of $L_2(\mathbb{R})$, and the characteristic function $\chi_{[-1,1]}(x)$ is an element of $L_2(\mathbb{R})$ but not an element of H_K. If we put $g(x) = \chi_{[-1,1]}(x)$, and consider the problem of finding a member of H_K very close to $g(x)$. It is not dificult to show that the problem

$$\inf_{f(x)\in H_K} \|f(x) - g(x)\|_{L_2(\mathbb{R})}$$

is unsolvable. In other words there is no best approximation of $g(x)$ in H_K. But if we put $g(x) = \chi_{[-1,1]}(x)$, and $L = I$ in the corollary of Theorem 3.1, then the problem

$$\inf_{f(x)\in H_K} \{\lambda\|f\|_K^2 + \|f(x) - g(x)\|_{L_2(\mathbb{R})}\} \tag{4.1}$$

is solvable for all $\lambda > 0$, and the best solution of (2.8) is

$$\begin{aligned} f\lambda^*(x) &= \int_{-\infty}^{\infty} g(\xi)\frac{1}{2\sqrt{\lambda(\lambda+1)}} \exp\left\{-\sqrt{\frac{\lambda+1}{\lambda}}|\xi - x|\right\} d\xi \\ &= \frac{1}{2\sqrt{\lambda(\lambda+1)}} \int_{-1}^{1} \exp\left\{-\sqrt{\frac{\lambda+1}{\lambda}}|\xi - x|\right\} d\xi. \end{aligned} \tag{4.2}$$

Hence for very small values of λ, we can assume that $f\lambda(x)$ are approximation of $g(x)$ in H_K. The following figure shows how $f\lambda(x)$ approaches to $g(x)$ as λ is made smaller.

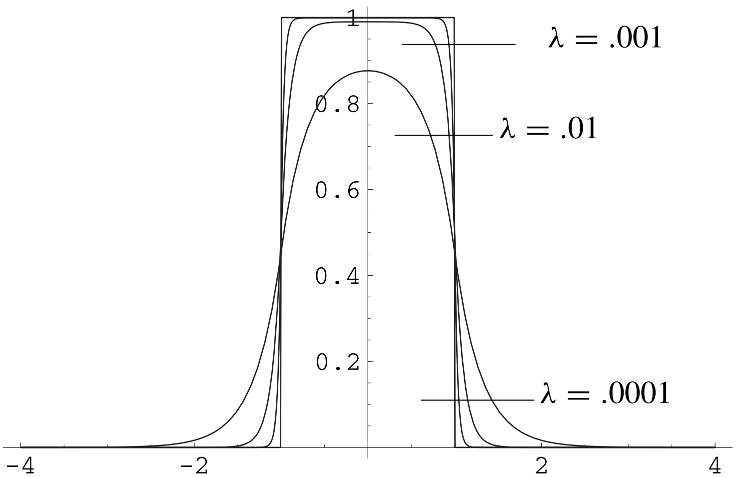

Figure 1: Graphs of $f\lambda^*(x)$ in (17) for $\lambda = .1$, $\lambda = .01$ and $\lambda = .001$.

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