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A Note On Modules Maps Over Finsler Modules

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Abstract

Let E and F be full Finsler modules over C*-algebras A and B respectively and $\psi : E \to F$ be a bijective bounded linear operator satisfies $\psi(ax) = \phi(a)\psi(x)$ and $\rho_B(\psi(x)) = \phi(\rho_A(x))$, for all $x \in E$ and $a \in A$, where $\phi : A \to B$ is a map. Then ϕ is a *-isomorphism. We also show that if ψ satisfies $\psi(ax) = \phi(a)\psi(x)$ for all $x \in E$ and $a \in A$, where $\phi : A \to B$ is a *-map, then ψ is a unitary operator iff it is a surjective isometry.

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1. Introduction

A (left) Hilbert C^*-module over a C^*-algebra A is a left A-module E equipped with A-valued inner product $\langle ., . \rangle$ which is a A-linear in the first and conjugate linear in the second variable such that E is a Banach space with the norm $\| x \| = \| \langle x, x \rangle \|^{\frac{1}{2}}$.

Finsler modules over C^*-algebras are generalization of Hilbert C^*-modules that first investigated in [5]. Let A be a C^*-algebra and A_+ denote the set of positive elements of a C^*-algebra A. Let E be a (left)module over C^*-algebra A and the map $\rho : E \to A_+$ satisfies the following conditions.

(i) the map $\|.\|_E : x \to \|\rho(x)\|$ is a Banach space norm on E; and

(ii) $\rho(ax)^2 = a\rho(x)^2 a^*$ for all $a \in A$ and $x \in E$.

Then E is called a Finsler module over C^*-algebra A.

A Finsler module over C^*-algebra A is said to be full if the linear span $\{\rho_A(x)^2; x \in E\}$ denoted by $\mathcal{F}(E)$ is dense in A.

A class of module maps of Hilbert C^*-modules that is called ϕ-morphisms class is introduced in the works of D. Bakic and B. Guljas [2]. M. Amyari and A. Niknam extended the definition of ϕ-morphisms over Finsler modules by the name of ϕ-homomorphisms and described some basic properties of such class of module maps of Finsler modules [1].

The aim of this paper is to continue the study of such module maps over Finsler modules. Firstly, in preliminaries we conclude some other properties of ϕ-homomorphisms over Finsler modules, in particular, in fullness condition. In the sequel, as main results, first we show that bijective bounded linear operators between full Finsler modules such that they have formal properties of a ϕ-homomorphisms (conditions (i), (ii) of Definition 2.1, uniquely determine a map ϕ as a *-isomorphism. In the remainder of the paper we will discuss about existence of ϕ-homomorphisms between two Finsler modules under some conditions (Theorem 3.5), and show that some module maps are automatically unitary operators (Theorem 3.6 and Corollary 3.7).

2. Preliminaries

Definition 2.1. let E and F be Finsler modules over C^*-algebra A and B respectively and $\phi : A \to B$ be a *-homomorphism of C^*-algebras. A linear operator $\psi : E \to F$ is said to be a ϕ-homomorphism of Finsler modules if the following conditions are satisfied:

(i) $\psi(ax) = \phi(a)\psi(x)$

(ii) $\rho_B(\psi(x)) = \phi(\rho_A(x))$

where $x \in E$ and $a \in A$. Recall that ψ is said to be module map if it satisfies in condition (i) (not necessarily in condition (ii)).

If E,F and G are Finsler modules over C^*-algebra A,B and C resp.; $\phi_1 : A \to B$, and $\phi_2 : B \to C$ are *-homomorphism of C^*-algebras, and $\psi_1 : E \to F$, and $\psi_2 : F \to G$ are ϕ_1- homomorphism and ϕ_2-homomorphism of Finsler modules resp; then it is straightforward to show that $\psi_2\psi_1 : E \to G$ is a $\phi_2\phi_1-$homomorphism of Finsler modules. Let E be a Finsler module over C^*-algebra A and I be an ideal of A (throughout this paper by an ideal we always mean a two-sided ideal) and also IE denoted by E_I be a closed linear span of the set $\{ax;\ a \in A, x \in E\}$. Clearly IE is a closed submodule of E and by Hewitt-Cohen factorization Theorem ([4] Theorem 4.1, and [6], Proposition 2.31) it is easy to show that $IE = \{ax;\ a \in A, x \in E\}$. Also IE is a Finsler module over I with respect to the norm Finsler $\rho_I(ax) = \rho_A(ax)$, since $\rho(ax)^2 = a\rho(x)^2a^* \in I$ whenever $a \in I$ and $x \in E$.

In the Following we show some properties of ϕ-homomorphisms.

Proposition 2.2. Let E and F be Finsler modules over C^*-algebra A and B respectively and $\psi : E \to F$ be a ϕ-homomorphism and I be an ideal of B. Then $\psi^{-1}(F_I) = E_{\phi^{-1}(I)}$.

Proof. In [Lemma 2.2, [1]] set $I = A$, then we conclude $AE = E$. Hence the following argument shows that $\psi^{-1}(F_I) = E_{\phi^{-1}(I)}$. Again By using [Lemma 2.2, [1]], for $x \in$

$E, a \in A$ we have

$$\begin{aligned} ax \in E_{\phi^{-1}(I)} &\leftrightarrow \rho_A(ax) \in \phi^{-1}(I) \leftrightarrow \phi(\rho_A(ax)) \in I \\ &\leftrightarrow \rho_B(\psi(ax)) \in I \leftrightarrow \psi(ax) \in F_I \\ &\leftrightarrow ax \in \psi^{-1}(F_I). \end{aligned}$$

■

If E is a full Finsler over C^*-algebra A, by Theorem 3.4(iii) of [1], $Img(\psi)$ is a full $Img(\phi)$-Finsler module. Conversely:

Theorem 2.3. Let E and F be Finsler modules over C^*-algebra A and B respectively and $\psi : E \to F$ be a ϕ-homomorphism. If $Img(\psi)$ is a full $Img(\phi)$-Finsler module, then with each of the following conditions, E is a full Finsler A-module.

(i) ϕ is injective,

(ii) $E_{ker(\phi)}$ is a $ker(\phi)$-full Finsler module.

Proof. Since $Img(\psi)$ is full Finsler $Img(\phi)$-module then for all $a \in A$, $\phi(a) = \lim \sum_{i \le k_n} \lambda_{i,n} \rho_B(\psi(x_{i,n}))^2$ for some $\lambda_{i,n} \in C$ and $x_{i,n} \in E$. Then $\phi\left(a - \lim \sum_{i \le k_n} \lambda_{i,n} \rho_A(x_{i,n})^2\right) = 0$ means $a - \lim \sum_{i \le k_n} \lambda_{i,n} \rho_A(x_{i,n})^2 \in ker(\phi)$. If ϕ be injective then $a = \lim \sum_{i \le k_n} \lambda_{i,n} \rho_A(x_{i,n})^2$ for all $a \in A$ shows that $A = \overline{\mathcal{F}(E)}$. Whenever $E_{ker(\phi)}$ is a $ker(\phi)$-full Finsler module for $I = ker(\phi)$, then $I = \overline{\mathcal{F}(E_I)}$ and for all $a \in A$ we have

$$a - \lim \sum_{i \le k_n} \lambda_{i,n} \rho_A(x_{i,n})^2 \in ker(\phi) = I = \overline{\mathcal{F}(E_I)} \subseteq \overline{\mathcal{F}(E)}.$$

Then $a \in \overline{\mathcal{F}(E)}$, since $\rho_A(x_{i,n})^2 \in \overline{\mathcal{F}(E)}$. Hence $A = \overline{\mathcal{F}(E)}$. ■

Corollary 2.4. Let let E and F be Finsler modules over C^*-algebra A and B respectively and $\psi : E \to F$ be a $\phi : A \to B$ homomorphism. If $Img(\psi)$ is a full $Img(\phi)$-Finsler module, then $\frac{E}{E_I}$ is full Finsler $\frac{A}{I}$-module, where $I = ker(\phi)$.

Proof. Define $\acute{\psi} : \frac{E}{E_I} \to F$ and $\acute{\phi} : \frac{A}{I} \to B$ with $\acute{\psi}(x + E_I) = \psi(x)$ and $\acute{\phi}(a + I) = \phi(a)$. By Theorem 3.3 of [1], $\acute{\psi}$ is a well defined $\acute{\phi}$-homomorphism. Then the result will be a straightforward of above theorem. ■

3. The Main Results

Lemma 3.1. Let E and F be Finsler and full Finsler modules over C*-algebras A and B respectively and ϕ_i $(i = 1, 2)$ are maps from A to B and surjective map $\psi : E \to F$ satisfies $\psi(ax) = \phi_i(a)\psi(x)$, $(i = 1, 2)$ for all $x \in E$ and $a \in A$. Then $\phi_1 = \phi_2$.

Proof. For every $x \in E$ and all $a \in A$ we have

$$\phi_1(a)\psi(x) = \psi(ax) = \phi_2(a)\psi(x).$$

Since ψ is surjective map by applying [1, Proof Theorem 3.2(iii)] the fullness of F implies that $\phi_1 = \phi_2$. ■

Theorem 3.2. Let E and F be full Finsler modules over C*-algebras A and B respectively and $\psi : E \to F$ be a bijective bounded linear operator satisfies $\psi(ax) = \phi(a)\psi(x)$ and $\rho_B(\psi(x)) = \phi(\rho_A(x))$, for all $x \in E$ and $a \in A$, where $\phi : A \to B$ is a map. Then ϕ is a *-isomorphism. Moreover, ϕ with these conditions is unique.

Proof. Let $x \in E$ and $a, b \in A$, we have

$$(\phi(a + b) - \phi(a) - \phi(b))\psi(x) = \psi((a + b)x) - \psi(ax) - \psi(bx)) = 0,$$

since ψ is linear. But ψ is surjective and F is full, then by [1, Proof Theorem 3.2(iii)]; $\phi(a + b) = \phi(a) + \phi(b)$. Similarly, $\phi(\lambda a) = \lambda\phi(a)$ and $\phi(ab) = \phi(a)\phi(b)$ for all $\lambda \in C$ and $a, b \in A$. Then ϕ is a homomorphism.

Assume that $\{a_n\}$ is a sequence in A such that $a_n \to 0$ and $\phi(a_n) \to b$ for $b \in B$. By [5, Lemma 2.2] E is a Banach A-module, then $a_n x \to 0$. Since ψ is continuous then $\psi(a_n x) \to 0$. Now condition (i) shows that $\phi(a_n)\psi(x) \to 0$. But $\phi(a_n) \to b$ then $\phi(a_n)\psi(x) \to b\psi(x)$. Therefore $b\psi(x) = 0$ for all $x \in E$. ψ is surjective and F is full, then $b = 0$. Since ϕ is linear, then it is continuous.

Let $a \in A$, by the fullness of E we may assume that $a = \lim_n u_n$, each u_n is of the form $u_n = \sum_{i \le k_n} \lambda_{i,n}\rho_A(x_{i,n})^2$ where $x_{i,n} \in E$ and $\lambda_{i,n} \in C$. By applying condition (ii), we have

$$\begin{aligned}
\phi(a^*) &= \phi\left(\lim_n u_n^*\right) = \phi\left(\lim_n \sum_{i \le k_n} \overline{\lambda_{i,n}}\rho_A(x_{i,n})^2\right) \\
&= \lim_n \sum_{i \le k_n} \overline{\lambda_{i,n}}\phi(\rho_A(x_{i,n}))^2 \\
&= \lim_n \sum_{i \le k_n} \overline{\lambda_{i,n}}\rho_B(\psi(x_{i,n}))^2 \\
&= \left(\lim_n \sum_{i \le k_n} \lambda_{i,n}\rho_B(\psi(x_{i,n})^2)\right)^* \\
&= \phi(a)^*.
\end{aligned}$$

Then ϕ is a *-homomorphism.

Now, by the hypothesis and [1, Theorems 3.2(iii) and 3.4(iv)] ϕ is a *-isomorphism from A to B. The last assertion clearly follows from Lemma 3.1. ■

Remark 3.3. Fullness condition can not be dropped in Theorems 3.2. For examples:

(a) Let $B = C[0,1]$, $A = E = \{f \in B : f(0) = 0\}$ and $F = \{f \in B : f(1) = 0\}$. Then E is a full Finsler A-module with respect to the norm Finsler $\rho_A(f) = |f|$ and F is a Finsler B-module with respect to the norm Finsler $\rho_B(f) = |f|$ which is not full. Let $\psi : E \to F$ with $\psi(f)(t) = f(1-t)$ for all $t \in [0,1]$ and $\phi : A \to B$ with $\phi(t) = \psi(t)$ for all $t \in [0,1]$. It is clear to show that ψ is a bijective bounded operator and $\psi(af) = \phi(a)\psi(f)$ and $\rho_B(\psi(f)) = \phi(\rho_A(f))$ for all $a \in A$ and $f \in E$. But ϕ is not $*$-isomorphism, since it is not surjective.

(b) Let A be an Von-Neumann algebra acting on a Hilbert space which has a central projection $p \neq 0, I$. Set $E = F = B = Ap$ and consider E and F as a Hilbert A-module and Hilbert B-module with the usual action and the inner products $< x, y >= xy^*$. Clearly E is not full A-module. Let ψ be inclusion map from E to F and $\phi : A \to B$, $\phi(a) = ap$. We have $\psi(ax) = \phi(a)\psi(x)$ and $\phi(< x, y >_A) =< x, y >_B$, but it is not one to one. Hence ϕ is not $*$-isomorphism.

Definition 3.4. Let A and B be C^*-algebras, E and F be Finsler modules over C^*-algebras A and B respectively. A linear operator $\psi : E \to F$ is said to be a unitary operator if there exists an injective homomorphism of C^*-algebras $\phi : A \to B$ such that ψ is a surjective ϕ-homomorphism.

In the following statements some cases are considered that condition (i) of definition of ϕ-homomorphism implies condition (ii).

Theorem 3.5. Let E and F be Finsler modules over C*-algebras A and B respectively and $\phi : A \to B$ is a *-isomorphism, then linear operator $\psi : E \to F$ which satisfies $\psi(ax) = \phi(a)\psi(x)$ for all $x \in E$ and $a \in A$, is a ϕ-homomorphism iff is isometry.

Proof. If ψ is ϕ-homomorphism then ψ is a isometry [1, Theorem 3.2(i)]. Conversely, let ψ is isometry, we have to show that ψ has the property $\rho_B(\psi(x)) = \phi(\rho_A(x))$. Let $x \in E$ and $b \in B$, then there exists $a \in A$ such that $\phi(a) = b$. we have

$$\begin{aligned}
||b\rho_B(\psi(x))||^2 &= ||b\rho_B(\psi(x))^2 b^*|| = ||\rho_B(b\psi(x))^2|| = ||\rho_B(\phi(a)\psi(x))^2|| \\
&= ||\rho_B(\psi(ax))^2|| = ||\psi(ax)||_F^2 = ||ax||_E^2 = ||\rho_A(ax)^2|| \\
&= ||\phi(\rho_A(ax)^2)|| = ||\phi(a\rho_A(x)^2 a^*)|| = ||\phi(a)\phi(\rho_A(x))||^2 \\
&= ||b\phi(\rho_A(x))||^2.
\end{aligned}$$

By [3, Lemma 3.4] it follows that $\rho_B(\psi(x)) = \phi(\rho_A(x))$, and so ψ is a ϕ-homomorphism. ■

Theorem 3.6. Let E and F be full Finsler modules over C*-algebras A and B respectively and $\phi : A \to B$ is a *-map, then linear operator $\psi : E \to F$ which satisfies

$\psi(ax) = \phi(a)\psi(x)$ for all $x \in E$ and $a \in A$, is an unitary operator iff is a surjective isometry.

Proof. If ψ is unitary operator then there exists an injective homomorphism of C^*-algebras $\varphi : A \to B$ such that ψ is a surjective φ-homomorphism. By Lemma 3.3 $\varphi = \phi$. Therefore by [1, Remark 3.6] ψ is an isometry.

Conversely, let ψ be surjective isometry. Firstly, we show that ϕ is injective. Let $\phi(a) = 0$. We have $\psi(ax) = \phi(a)\psi(x) = 0$ for all $x \in E$. Since ψ is one to one, it follows that $ax = 0$ for all $x \in E$. Since E is full [1, Proof of Theorem 3.2(iii)] implies that $a = 0$. Therefor ϕ is injective.

As Theorem 3.2 we can show that ϕ is a homomorphism and since ϕ is a *-map then it is a *-homomorphism. Hence ϕ is isometry and $\phi(A_+) \subseteq B_+$.

Now we show that condition (ii) of definition ϕ-homomorphism is held. Define $\rho'_B : F \to B$ with $\rho'_B(\psi(x)) = \phi(\rho_A(x))$. It is clear that ρ'_B make E a finsler module. We prove that $\rho'_B = \rho_B$. To do this, in view of [5, Corollary 3.7] we have to ρ'_B and ρ_B induce the same norm $||.||_F$. Let $x \in E$,

$$\begin{aligned} ||\psi(x)||_{\rho_B} &= ||x||_{\rho_A} = ||\rho_A(x)|| \\ &= ||\phi(\rho_A(x))|| = ||\rho'_B(\psi(x))|| \\ &= ||\psi(x)||_{\rho'_B}. \end{aligned}$$

Since ψ is onto then ρ'_B and ρ_B induce the same norm $||.||_F$. Therefore, $\rho_B(\psi(x)) = \phi(\rho_A(x))$. Thus ψ is a ϕ-homomorphism. By [1, Theorem 3.4(iv)] ψ is surjective. Therefore, by the definition ψ is unitary operator. ■

Corollary 3.7. Let E and F be full Finsler modules over C^*-algebra A and $\psi : E \to F$ be a surjective isometry A-linear map then,

(i) ψ is unitary operator,

(ii) Identity map is only homomorphism which makes ψ to a ϕ-homomorphism (unitary).

Proof. These are a straightforward of Theorem 3.6 and Lemma 3.1. ■

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