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On Generalized Stirling Numbers and Polynomials

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Abstract

The object of this article is to present a generalization of stirling numbers and polynomials which were studied in a number of earlier work on the subject due to their importance for possible applications in certain problems arising in science and engineering (like curve fitting, coding theory, signal processing etc.). We prove that are result concerned the generalized stirling numbers are the consequence of the result of Toscano and Chak. The new explicit expressions for generalized stirting numbers are also given. The results obtained are of general character and include the investigations carried out by several authors including Riordan Singh, Sinha and Dhawan, Shrivastava etc.

Key words: Stirling numbers of second order, generalized stirling numbers, & polynomials.

1. Introduction

The work of Riordan [7] (p. 90 et seq.), we denote by $S(n,m)$ the stirling numbers of second kind, defined by

$$S(\mathrm{n},\mathrm{m}) = \frac{1}{m!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}j^n = \frac{(-1)^m}{m!}\Delta^m O^n . \tag{1}$$

So that
$$S(\mathrm{n},0) = \begin{cases} 1 & (n=0) \\ 0 & (n\in N\{1,2,3....\}) \end{cases} \tag{2}$$

And $S(\mathrm{n},1) = S(\mathrm{n},\mathrm{n}) = 1$ and $S(\mathrm{n},\mathrm{n}-1) = \binom{n}{2}$

The corresponding polynomials so called single variable Bell polynomials (sec [6], [12]), are defined by

$$A_n(x) = \sum_{m=0}^{n} S(n,m)x^m , \tag{3}$$

Singh [19], Sinha and Dhawan [20] and Shrivastava [18], studied the generalized stirling numbers and polynomials defined, respectively, by

$$S^{(\alpha)}(n,m,r) = \frac{(-1)^m}{m!}\sum_{j=0}^{m}(-1)^j \binom{m}{j}(\alpha + rj)^n \tag{4}$$

$$\text{and } T_n^{(\alpha)}(x,r,-p) = \sum_{m=0}^{n} S^{(\alpha)}(n,m,r)p^m x^{rm} \tag{5}$$

It is easily verified from [10] that

$$T_n^{(\alpha)}(x,r,-p) = x^{-\alpha}e^{px^r}(xD)^n x^{\alpha}e^{px^r}$$

i. e. the generalized Truesdell polynomials [14].
Singh's generalization was motivated by the generalization of Hermite polynomials of Gould-Hopper [5] given by

$$H_n^{(r)}(x,\alpha,p) = (-1)^n x^{-\alpha}e^{px^r}D^n(x^{\alpha}e^{-px^r})$$

In this article, we will study the following new generalized stirling numbers and polynomials in the following general form

$$S^{(\alpha,\beta,\lambda)}(n,m,r) = \frac{(-1)^m}{m!}\sum_{j=0}^{m}\binom{m}{j}(\alpha + rj)^{(\beta-\lambda,n)} \tag{6}$$

$$\text{and } T_n^{(\alpha,\beta,\lambda)}(x,r,-p) = x^{n(\beta-\lambda)}\sum_{m=0}^{n} S^{(\alpha,\beta,\lambda)}(n,m,r)p^m x^{rm} \tag{7}$$

$$\text{where } a^{(\beta-\lambda,n)} = \left(\frac{a}{\beta-\lambda}\right)_n (\beta-\lambda)^n$$

evidently, when $\beta \to \lambda$, equation (6) and (7) would reduce to (4) and (5) respectively, which in turns, will yield (1) and (3) respectively for $r-1 = \alpha = 0$.
In this paper we prove that our result by using the fundamental result of Toscano [14] and Chak [1].
The new explicit expression for our numbers are also given.

Recurrence formula & proof

The recurrence formula as well as proof of our result can be easity derived from the result of A. M. Chak [1].
Starting with the following notation.

$$a^{(\beta-\lambda,n)} = \left(\frac{a}{\beta-\lambda}\right)_n (\beta-\lambda)^n = a(a+\beta-\lambda)\ldots\{a+(\beta-\lambda)(n-1)\}$$

we see that $(\alpha+rj)^{(\beta-\lambda,n)} = r^n \left(\frac{\alpha}{r} + j\right)^{\left(\frac{\beta-\lambda}{r},n\right)}$

and so, from (6), we have

$$S^{(\alpha,\beta,\lambda)}(n,m,r) = \frac{(-1)^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j} r^n \left(\frac{\alpha}{r}+j\right)^{\left(\frac{\beta-\lambda}{r},n\right)} = r^n A_{n,m}^{\left(\frac{\alpha}{r},\frac{\beta-\lambda+1}{r}\right)}$$

where $A_{n,m}^{(\alpha,\lambda)}$ are numbers defined by A. M. Chak [1] in the following way

$$(x^{\lambda}D)^n = x^{(n-1)\lambda}\sum_{j=0}^{n} A_{n,i}^{(\alpha,\lambda)} x^{i+\alpha} D^i x^{-\alpha}$$

so, using proved equations and the recurrence from [1]

$$A_{n+1,m}^{(\alpha,\lambda)} = A_{n,m-1}^{(\alpha,\lambda)} + (n\lambda - n + \alpha + m) A_{n,m}^{(\alpha,\lambda)}$$

we get a recurrence formula for (6)

$$S^{(\alpha,\beta,\lambda)}(n+1,m,r) = rS^{(\alpha,\beta,\lambda)}(n,m-1,r) + (n\beta - n\lambda + \alpha + rm)S^{(\alpha,\beta,\lambda)}(n,m,r) \quad (8)$$

for polynomial (7), we get from [1] easily

$$T_n^{(\alpha,\beta,\lambda)}(x,r,-p) = r^n x^{n(\beta-\lambda)} G_{n,\frac{\beta-\lambda+1}{r}}^{(\alpha/r)}(-px^r)$$

where $G_{n,\lambda}^{(\alpha)}(x)$ are a class of polynomials from [1].

Explicit expression for $S^{(\alpha,\beta,\lambda)}(n,m,r)$

In this section, we use the results from Caki`c [ 16 ] (see theorem 4 and comment by L. Carlitz) and milovanovic` and Cakic` [4] (see theorem 3).
Applying Carlitz's [8] extension of a result of wang [17].
In this section, we prove the theorem about new explicit expressions for numbers $S^{(\alpha,\beta,\lambda)}(n,m,r)$.

Theorem: We have

$$S^{(\alpha,\beta,\lambda)}(n,m,r) = r^n \sum \binom{\alpha/r}{i_0} \prod_{s=1}^{n-1} \binom{\alpha/r + s\left(\frac{\beta-\lambda}{r}+1\right) - S_s}{i_s} \quad (9)$$

where $S_s = \sum_{t=0}^{s-1} i_t$. The summation on the right in above sums is overall $i_0,\ldots, i_{n-1} \in \{0, 1\}$, such that $i_0+i_1+\ldots+i_{n-1} = n-m$

proof: Introduce the notation.

$$C(i_0, \ldots, i_{n-2}) = \binom{\alpha/r}{i_0}\prod_{s=1}^{n-2}\binom{\alpha/r + s\left(\frac{\beta-\lambda}{r}+1\right) - S_s}{i_s} \tag{10}$$

For $i_{n-1} \in \{0, 1\}$ we get from (9)

$$S^{(\alpha,\beta,\lambda)}(n,m,r) =$$

$$r^n . \sum_{S_{n-1}=(n-1)-(m-1)} C(i_0,\ldots,i_{n-2}) + r^n\left(\frac{\alpha}{r} + (n-1)\left(\frac{\beta-\lambda}{r}+m\right)\right)\sum_{S_{n-1}=(n-1)-m} C(i_0,\ldots,i_{n-2})$$

i. e.

$$S^{(\alpha,\beta,\lambda)}(n,m,r) =$$

$$rs^{(\alpha,\beta,\lambda)}(n-1,m-1,r) + \{(n-1)(\beta-\lambda) + \alpha + rm\}S^{(\alpha,\beta,\lambda)}(n-1,m,r)$$

for m=0 and m=n, we have

$$S^{(\alpha,\beta,\lambda)}(n,0,r) = r^n \prod_{s=0}^{n-1}\left(\frac{\alpha}{r} + s.\frac{\beta-\lambda}{r}\right) = \alpha^{(\beta-\lambda,n)}$$

$$S^{(\alpha,\beta,\lambda)}(n,n,r) = r^n$$

this complete the proof.

Remarks : In above theorem we use the convention if α=0 then i_0=0.
Notice that for bigger values of m (m>[n/2]) this formula are simpler than classical expression (6). For example if we take m=n-1, then from (6), we find.

$$S^{(\alpha,\beta,\lambda)}(n,n-1,r) = \frac{(-1)^{n-1}}{(n-1)!}\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(\alpha+rj)^{(\beta-\lambda,n)}$$

while representations (10) give

$$S^{(\alpha,\beta,\lambda)}(n,n-1,r) = n\alpha r^{n-1} + \binom{n}{2}\left\{\left(\frac{\beta-\lambda}{r}\right)+1\right\}r^n$$

2. Particular Cases

(i) Through our generalized stirling numbers and polynomials, we can deduce all the generalized stirling numbers and polynomials and their special cases.

(ii) Some important special cases of our numbers $S^{(\alpha,\beta,\lambda)}(n,m,r)$ are "weighted stirling numbers of the second kind" ([10] [11]). R (n,m,λ) = $S^{(\lambda)}(n,m,1) = S^{(\lambda,1,1)}(n,m,1)$

(iii) "Degenerate stirling numbers of the second kind [9] $S(n,m/\theta) = S^{(0,1-\theta,1)}(n,m,1)$.

(iv) Degenerate weighted stirling numbers of the second kind" [3] . $S(n,m,\lambda/\theta) = S^{(\lambda,1-\theta,1)}(n,m,1)$.

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