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Existence of Solution for a Boundary Value Problem of Fractional order

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Abstract

In this paper, we are concerned with the existence of positive solutions for the nonlinear fractional differential:

$$D_t^{\alpha} u(t) = f(t, u(t)), \quad 0 < t < 1,\ 1 < \alpha \le 2,$$
$$\beta u(0) - \gamma u'(0) = \delta, \quad \lambda u(1) + \upsilon u'(1) = \mu.$$

where D_t^{α} is the Caputo's fractional derivative, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. Our analysis relies on a nonlinear alternative of Leray-Schauder fixed point theorem.

Keywords: Caputo's fractional derivative, boundary-value problem, positive solution, fixed-point theorem.

1. Introduction

Differential equations of fractional order, or fractional differential equations, in which an unknown function is contained under the operation of a derivative of fractional order, have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, etc. It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations in terms of special functions. Recently, there are some papers deal with the existence and multiplicity of solution (or positive solution) of nonlinear initial fractional differential equation by the use of techniques of nonlinear analysis (fixed-point theorems, Leray–Shauder theory, etc.). Bai and Lü [2] studied the existence and multiplicity of positive solutions of nonlinear fractional differential equation boundary value problem:

$$D_t^{\alpha}u(t) = f(t,u(t)), \quad 0 < t < 1,\ 1 < \alpha \le 2, \qquad (1)$$
$$u(0) = u(1) = 0.$$

where D_t^{α} is the Standard Riemann- Liouville fractional derivative. Zhang [5] considered the existence of solution of nonlinear fractional differential equation boundary value problems involving Caputo's derivative.

$$D_*^{\alpha}u(t) = f(t,u(t)), \quad 0 < t < 1,\ 1 < \alpha \le 2, \qquad (2)$$
$$u(0) = \alpha \ne 0, \quad u(1) = \beta \ne 0.$$

In another paper, by using fixed point theorem on cones, Zhang [6] studied the existence and multiplicity of positive solutions of nonlinear fractional differential equation boundary value problem:

$$D_t^{\alpha}u(t) = f(t,u(t)), \quad 0 < t < 1,\ 1 < \alpha \le 2, \qquad (3)$$
$$u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0$$

where D_*^{α} is the Caputo's fractional derivative. In this article, we consider the existence of solution of the following boundary value problem of fractional order:

$$D_*^{\alpha}u(t) = f(t,u(t)), \quad 0 < t < 1,\ 1 < \alpha \le 2, \qquad (4)$$

$$\beta u(0) - \gamma u'(0) = \delta, \quad \lambda u(1) + \upsilon u'(1) = \mu. \qquad (5)$$

where D_*^{α} is the Caputo's fractional derivative and $\beta, \gamma, \delta, \lambda, \nu, \mu \ge 0$.

2. Preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions can be found in the recent literature.

Definition 2.1

The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : [0,\infty) \to \Re$ is given by:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds\,, \qquad (6)$$

Definition 2.2

The Caputo's fractional derivative of order $\alpha \phi 0$ can be written as:

$$D_*^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \qquad n = [\alpha] + 1 \qquad (7)$$

Remark 2.1. As a basic example, for $\lambda > -1$,

$$D_*^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}. \tag{8}$$

From the definition of Caputo's derivative and Remark 2.1, we can obtain the statement.

Lemma 2.1. Let $\alpha > 0$, then we have [6]

$$I^\alpha D_*^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \dots\dots\dots + c_n t^n, \tag{9}$$

for some $c_i \in \Re,\ i = 0,1,\dots\dots,n$

Theorem 2.1. [Agarwal et. al [1] and O'Regan [3]] Assume that *U* is a relatively open subset of convex set *K* in Banach space *E*. Let $N : \overline{U} \to K$ be a compact map with $0 \in U$. Then either

(i) *N* has a fixed point in $\overline{U}$; or

(ii) There is a $u \in U$ and a $\lambda \in (0,1)$ such that $u = \lambda N u$.

The Integral Equation

At first, we will find the solution *u(t)* for the problem:

$$D_*^\alpha u(t) = y(t), \quad 0 < t < 1,\ 1 < \alpha \le 2, \tag{10}$$

with

$$\beta u(0) - \gamma u'(0) = \delta, \quad \lambda u(1) + \upsilon u'(1) = \mu. \tag{11}$$

Applying Laplace transforms to equation (10), then we have:

$$s^\alpha \overline{u}(s) - u(0) s^{\alpha-1} - u'(0) s^{\alpha-2} = y(s), \tag{12}$$

where $\overline{u}(s)$ and *y(s)* is the Laplace transform of $u(t)$ and $y(t)$ respectively, so

$$\overline{u}(s) = \frac{u(0)}{s} + \frac{u'(0)}{s^2} + \frac{1}{s^\alpha} y(s), \tag{13}$$

therefore, by the inverse Laplace transform, we obtain

$$u(t) = u(0) + u'(0)\, t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds. \tag{14}$$

Using the boundary condition (11), we obtain the final form of $u(t)$ as:

$$u(t)=A+Bt+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds-\frac{\lambda\gamma}{\rho}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds$$
$$-\frac{\upsilon\gamma}{\rho}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds-\frac{\lambda\beta t}{\rho}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds-\frac{\nu\beta t}{\rho}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds, \quad (15)$$

where $A=\dfrac{\lambda\delta+\nu\delta+\gamma\mu}{\rho}$, $B=\dfrac{\beta\mu}{\rho}-\dfrac{\lambda\delta}{\rho}$ and $\rho=\lambda\beta+\nu\beta+\lambda\gamma$.

3. Main Results

Consider the family of problems:

$$\begin{aligned}&D_*^{\alpha}u(t)=h\ f(t,u(t)),\quad 0<t<1,\ 1<\alpha\le 2\\ &\beta u(0)-\gamma u'(0)=\delta,\quad \lambda u(1)+\upsilon u'(1)=\mu.\end{aligned}, \quad (16)$$

where $h\in(0,1)$. Hence (16) is equivalent to the integral equation

$$u(t)=A+Bt+h\left[\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds-\frac{\lambda\gamma}{\rho}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds\right.$$
$$-\frac{\upsilon\gamma}{\rho}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s))ds-\frac{\lambda\beta t}{\rho}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds \quad (17)$$
$$\left.-\frac{\nu\beta t}{\rho}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s))ds\right].$$

Defining $T:X\to X$ as:

$$Tu(t)=A+Bt+h\left[\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds-\frac{\lambda\gamma}{\rho}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds\right.$$
$$-\frac{\upsilon\gamma}{\rho}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s))ds-\frac{\lambda\beta t}{\rho}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds \quad (18)$$
$$\left.-\frac{\nu\beta t}{\rho}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s))ds\right].$$

where $X=C[0,1]$ is the Banach space endowed with the supper norm. We have the following result for operator T.

Lemma 4.1

Assume that $f:[0,1]\times\Re\to\Re$ is continuous function, then T is completely continuous operator.

Proof: It is easy to see that T is continuous. For $u \in M = \{u \in X; \|u\| \le l, l > 0\}$, we obtain

$$|T\,u(t)| = \left| \begin{array}{r} A + B\,t + h\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds - \frac{\lambda\,\gamma}{\rho} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds \right. \\ - \frac{\upsilon\gamma}{\rho} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds - \frac{\lambda\,\beta\,t}{\rho} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds \\ \left. - \frac{\nu\beta\,t}{\rho} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds \right) \end{array} \right|$$

$$\le A + B + h\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds - \frac{\lambda\,\gamma}{\rho} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds \right.$$

$$- \frac{\upsilon\gamma}{\rho} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds - \frac{\lambda\,\beta\,t}{\rho} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds$$

$$\left. - \frac{\nu\beta\,t}{\rho} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds \right)$$

$$\le A + B + h\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds - \frac{\lambda\,\gamma}{\rho} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds \right.$$

$$- \frac{\upsilon\gamma}{\rho} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds - \frac{\lambda\,\beta\,t}{\rho} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds$$

$$\left. - \frac{\nu\beta\,t}{\rho} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds \right)$$

$$\le A + B + h\left(L\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \frac{\lambda\,\gamma\,L}{\rho} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \frac{\upsilon\gamma\,L}{\rho} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \right.$$

$$\left. + \frac{\lambda\,\beta\,L}{\rho} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \quad + \frac{\nu\beta\,L}{\rho} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \right)$$

$$\le A + B + \frac{L}{\Gamma(\alpha+1)} + \frac{\lambda\gamma L}{\rho\Gamma(\alpha+1)} + \frac{\lambda\beta L}{\rho\Gamma(\alpha+1)} + \frac{\nu\gamma L}{\rho\Gamma(\alpha)} + \frac{\nu\beta L}{\rho\Gamma(\alpha)},$$

where $L = \max\limits_{0\le t\le 1, \|u\|\le 1} |f(t,u(t)| + 1$, so $T(M)$ is bounded. Next we shall show the equicontinuity of $\overline{T(M)}$. $\forall u \in M, \forall \varepsilon > 0, t_1 < t_2 \in [0,1]$.

Let $\eta < \left\{ \dfrac{\varepsilon\rho}{3|\beta\mu-\lambda\delta|}, \left(\dfrac{\varepsilon\Gamma(\alpha)}{6L}\right)^{\frac{1}{\alpha-1}}, \dfrac{\varepsilon\rho\Gamma(\alpha+1)}{3(\lambda+\nu)\beta L} \right\}$, then when $t_2 - t_1 < \eta$, we have

$$|Tu(t_2) - Tu(t_1)| =$$

$$\left| \begin{array}{l} B(t_2-t_1) + h\displaystyle\int_0^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^s f(\tau,u(\tau))\,d\tau\,ds - h\int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^s f(\tau,u(\tau))\,d\tau\,ds \\ -\dfrac{\lambda h\beta}{\rho}(t_2-t_1)\displaystyle\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds - \frac{\nu h\beta}{\rho}(t_2-t_1)\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds \end{array} \right|$$

$$\leq \left|\frac{\beta\mu}{\rho} - \frac{\lambda\delta}{\rho}\right|(t_2-t_1) + \frac{L}{\Gamma(\alpha)}\left[t_2^{\alpha-1} + 2(t_2-t_1)^{\alpha-1} - t_1^{\alpha-1}\right] + \frac{\lambda\beta L(t_2-t_1)}{\rho\Gamma(\alpha+1)} + \frac{\nu\beta L(t_2-t_1)}{\rho\Gamma(\alpha+1)}$$

$$\leq \left|\frac{\beta\mu}{\rho} - \frac{\lambda\delta}{\rho}\right|\eta + \frac{2L}{\Gamma(\alpha)}\eta^{\alpha-1} + \frac{(\lambda+\nu)\beta L\eta}{\rho\Gamma(\alpha+1)}$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.$$

Thus $\overline{T(M)}$ is equicontinuous. The Arzela-Ascoli theorem implies that the operator T is completely continuous.

Theorem 4.1
Assume that $f : [0,1] \times R \to R$ is continuous function, and there exist constants $\phi = \dfrac{\lambda\gamma + \nu\gamma\alpha + \lambda\beta + \upsilon\beta\gamma + \rho}{\rho\,\Gamma(\alpha+1)}$, $0 < c_1 < \dfrac{\Gamma(\alpha+1)}{3\phi}$, $c_2 > 0$, $0 < \theta \leq 1$, such that $|f(t,u(t)| \leq c_1|u|^{\theta} + c_2$ for all $t \in [0,1]$. Then the boundary value problem (4), (5) has a solution.

Proof: Following [5], we will apply the nonlinear alternative theorem to prove that T has one fixed point.

Let $\Omega = \{u \in X; \|u\| < R\}$, be open subset of *X*, where $R > \left\{3(|A|+|B|), \dfrac{3\phi c_2}{\Gamma(\alpha+1)}\right\}$. We suppose that there is a point $u \in \partial\Omega$ and $\lambda \in (0,1)$ such that $u = Tu$. So. For $u \in \partial\Omega$, we have:

$$
|T\,u(t)| = \left| \begin{array}{l}
A + Bt + h\left(\displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds - \frac{\lambda\gamma}{\rho}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds \right. \\
\quad - \dfrac{\upsilon\gamma}{\rho}\displaystyle\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds - \frac{\lambda\beta t}{\rho}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds \\
\qquad \left. - \dfrac{\nu\beta t}{\rho}\displaystyle\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds \right)
\end{array} \right|
$$

$$
\begin{aligned}
&\le A + B + \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds + \frac{\lambda\gamma}{\rho}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds \right. \\
&+ \frac{\upsilon\gamma}{\rho}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds + \frac{\lambda\beta}{\rho}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds \\
&\qquad \left. + \frac{\nu\beta}{\rho}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds \right) \\
&\le A + B + \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (c_1|u(s)|^{\theta} + c_2)ds + \frac{\lambda\gamma}{\rho}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} (c_1|u(s)|^{\theta} + c_2)ds + \right. \\
&+ \frac{\upsilon\gamma}{\rho}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} (c_1|u(s)|^{\theta} + c_2)ds + \frac{\lambda\beta}{\rho}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} (c_1|u(s)|^{\theta} + c_2)ds \\
&\qquad \left. + \frac{\nu\beta}{\rho}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} (c_1|u(s)|^{\theta} + c_2)ds \right) < \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R,
\end{aligned}
$$

which implies that $\|T\| \ne R = \|u\|$, that is a contraction. Then the nonlinear alternative theorem implies that T has a fixed point $u \in \overline{\Omega}$, that is, problem (4), (5) has a solution $u \in \overline{\Omega}$.

Finally, we give an example to illustrate the results obtained in this paper.

Example For the boundary value problem

$$
\begin{aligned}
&D_t^{1.5}u(t) = \frac{0.1u^{\theta} + 1}{u^2 + 7}, \quad 0 < t < 1, \\
&u(0) - u'(0) = 0, \quad u(1) + u'(1) = 0
\end{aligned}
\quad . \tag{19}
$$

We apply theorem 4.1 with $\alpha = 1.5,\ \beta = 1,\ \gamma = 1,\ \lambda = 1, \nu = 1,\ \delta = 0,\ \mu = 0.$ Then we have $\rho = 1,\ \phi = 4.1374$ and $c_1 = 0.1 < \dfrac{\Gamma(1+\alpha)}{3\phi}$. We conclude that problem (19) has a solution.

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