Advances in Algebra
ISSN 0973-6964 Volume 11, Number 1 (2018), pp. 19-28

http://www.ripublication.com

On Right Centralizers of Semiprime Semirings

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Abstract

Let S be a semiprime semiring with centre Z. A mapping $h: S \to S$ is called centralizing on S if $[x, h(x)] \in Z(S)$ for all $x \in S$. We show that every centralizing right centralizer of a semiprime semiring is a centralizer. We also show that if T and H are Right centralizers of a semiprime semiring S satisfying $xT(x) + H(x)x \in Z(S)$ for all $x \in S$. Then both T and H are centralizers of S.

AMS Subject Classification : 16Y60

Keywords: semiring, semiprime semiring, left(right) centralizer, centralizer, commuting mappings, centralizing mappings, Jordan subring..

1. INTRODUCTION AND PRELIMINARIES

Without giving the formal definition of semiring, semirings first appear implicitly in Richard Dedekind's supplement 11. The algebraic structure of semiring was introduced by H.S. Vandiver in 1934. This system consisting of a nonempty set R

with two binary operations, addition (usually denoted by "+") and multiplication (usually denoted by ".") such that the multiplication is distributive over addition. A natural example of semiring which is not a ring is set of Natural numbers N, under usual addition and multiplication of numbers. Numerous researchers made good work on centralizers of rings. Recently many authors established the study of semirings and semiprime rings. Herstein and Neuman investigated centralizers in Rings which arises in the study of rings with involution. Zalar [1] worked on centralizers of semiprime rings. M.F.Hoque and A.C.Paul [6,7] studied on centralizers of semiprime Γ-rings . Postner initiated the study of centralizing mappings in Rings. M.Breser and J. Vukman [2,3] established very striking results on centralizing mappings in prime rings. H.E. Bell, W.S.Martindale III [11] proved some remarkable results on centralizing mappings of semiprime Rings . Golan [9] discussed the notion of semirings and their applications. But not gave more brief explanations about centralizers of semirings. Chandramouleeswaran and Tiruveni [4] studied derivations on semiring. M.S. Sammam and M.A.Chaudhry [12] proved that, If two left centralizers T and S of a semiprime ring R satisfying $T(x)x + xS(x) \epsilon Z(R)$, then both T and S are centralizers. K.K.Day and A.C.Paul [8] made the general frame work on left centralizers of semiprime Gamma ring. The purpose of this paper is to propose the notion of Right centralizers of semiprime semiring and proved some simple results. The results in this paper for Right centralizers are also true for left centralizers because of left-right symmetry.

Definition 1.1 : *A semiring is a non empty set S followed with two binary operations addition and multiplication such that $(S,+)$ is a semigroup $(S,.)$ is a semigroup Multiplication distributes over addition from either side.*

Definition 1.2: *The semiring $(S,+,.)$ is said to be semiring with additive identity* 0 *such that $x + 0 = x = 0 + x$ for all $x \epsilon S$ and addition is commutative. A semiring $(S,+,.)$ is said to have an identity element* 1, *if* $1 \neq 0$ *such that* $1.x = x = x.1$ *for all $x \epsilon S$ and multiplication is commutative. Semiring $(S,+,.)$ is said to be commutative if both $(S+)$ and $(S,.)$ are commutative.*

Definition 1.3 : *A Semiring S is said to be prime if $xSy = 0$ implies $x = 0$ or $y = 0$ for all $x, y \epsilon S$*
A Semiring S is said to be semiprime if $xSx = 0$ implies $x = 0$ for all $x \epsilon S$.
A semiring S is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \epsilon S$.

Definition 1.4: *A semiring S is said to be commutative semiring if $x\,y = yx$ for all $x\,, y \in S$. Then the set $Z(s) = \{x \in S,\ xy = yx$ for all $y \in S\}$ is called the centre of the semiring S.*

Definition1.5: *For any fixed $a \in S$ the mapping $T(x) = ax$ is a left centralizer and $T(x) = xa$ is a right centralizer. An additive mapping $T: S \to S$ is a left (Right) centralizer if $T(xy) = T(x)y$, $((T(xy) = xT(y))$ for all $x\,, y \in S$.*

Definition 1.6: *If S is a semiring then $[x, y] = xy + y'x$ is known as the commutator of x and y. The following are the basic commutator identities. $[xy, z] = [x, z]y + x[y, z]$ and*

$[x, yz] = [x, y]z + y[x, z]$ *for all $x\,, y, z \in S$.*

According to [13] $for\ all\ a, b \in S$ we have

$(a + b)' = a' + b'$

$(ab)' = a'b = ab'$

$a'' = a$

$a'b = (a'b)' = (ab)'' = ab$

Also the following implication is valid.

$a + b = o\ implies\ a = b'$ and $a + a' = 0$

Definition 1.7: *A mapping h: $S \to S$ is called centralizing (skew centralizing) if $[h(x), x] \in Z(S)$. $(h(x)x + xh(x) \in Z(S))$ for all $x \in S$. A mapping h of a semiring S into itself is said to be commuting if $[h(x), x] = 0$ and skew commuting if $(h(x)x + xh(x) = 0)$. Obviously every commuting(skew commuting) mapping h: $S \to S$ is centralizing (skew centralizing). We recall if h: $S \to S$ is commuting then $[h(x), y] = [x, h(y)]$ for all $x, y \in S$. A mapping h: $S \to S$ is called central if $h(x) \in Z(S)$ for all $x \in S$.*

Definition 1.8: *Let A be a subset of S. If $xy + yx \in A$ for all $x, y \in A$ then A is called a Jordan subring of S.*

Notation 1.1: *We consider semiring with absorbing Zero and with identity* $1 \neq 0$.

2 MAIN RESULTS

Theorem 2.1: *Let S be a 2- torsion free semiprime semiring and A be a Jordan subring of S. If an additive mapping H of S into itself is centralizing on A, then H is commuting on A.*

Proof: By the assumption $[H(x), x] \epsilon Z(S)$. The linearization gives $[H(x), y] + [H(y), x] \epsilon Z$ for all $x, y \epsilon A$. In particular if $y = x^2$ one obtains that

$2[H(x), x]x + [H(x^2), x] \epsilon Z$ for all $x \epsilon A$. (1)

On the other hand we have $[H(x^2), x^2] \epsilon Z$ for all $x \epsilon A$

That is $[H(x^2), x]x + x\,[H(x^2), x] \epsilon Z$ (2)

Let $x \epsilon A$ be a fixed element and let $z = [H(x), x] \epsilon Z$, $a = [H(x^2), x]$ in (1) we obtain

$[H(x), 2zx + a] = 0$. We are forced to show that $z = 0$.

$[H(x), 2zx] + [H(x), a] = 0$

$2z[H(x), x] + 2[H(x), z]x + [H(x), a] = 0$

$[H(x), a] = 2z'^2$ (3)

Now using (2), one can write $[H(x), ax + xa] = 0$ and apply (3) we get

$[H(x), a]x + a[H(x), x] + [H(x), x]a + x[H(x), a] = 0$

$2za + 4z'^2x = 0$

$za = 2z^2x$

multiplying (3) by z we obtain $z[H(x), a] = 2zz'^2$

$[H(x), za] + [H(x), z]a' = 2zz'^2$

Substituting $za = 2z^2x$ in the above

$[H(x), 2z^2x] = 2zz'^2$

$2\{[H(x), z^2]x + z^2[H(x), x]\} = 2zz'^2$

Adding $2z^3$ on both sides, we get $2z^3 + 2z^3 = 0$

$2z^3 = 0$. S is 2 torsion free semiprime semiring $z^3 = 0$, Since the centre of a semiprime semiring contains no non zero nilpotent elements. This leads us to conclude that $z = 0$. This proves the theorem.

Theorem 2.2: *Let S be a semiprime semiring. If $T: S \to S$ be a centralizing Right Centralizer. Then T is commuting.*

Proof: If S is 2 torsion free, then by taking $A = S$ in theorem (2.1) it follows immediately that T is commuting. If S is not a 2 torsion free semiprime semiring, then

$2[x, T(x)] = 0$ for all x ϵS (4)

Substituting $x + y$ for x, we arrive at $2[x, T(y)] + 2[y, T(x)] = 0$

$[x, T(y)] = [y, T(x)]'$ for all $x, y \epsilon S$. (5)

linearising the assumption $[x, T(x)] \epsilon Z(s)]$ and using (5) we obtain

$[x, T(y)] + [y, T(x)] \epsilon Z(s)$ (6)

$[([x, T(y)] + [y, T(x)]), x] = 0 \; for \; all \; x, y \epsilon S$

$[x, T(y)]x + [y, T(x)] + x'[x, T(y)] + x'[y, T(x)] = 0$

$[x, T(y)]x + [y, T(x)]x + x[y, T(x)] + x[x, T(y)] = 0$ (7)

From assumption we found the result $[x, T(x)]y = y[x, T(x)]$

Applying the above result in relation (4) and is reached at

$[x, T(x)]y + y[x, T(x)] = 0$ (8)

Adding expressions (7) and (8) then arranged, we get

$[xy + yx, T(x)] + [x^2, T(y)] = 0$ (9)

Now using xy for y in the above relation we arrive at

$[xxy + xyx, T(x)] + [x^2, T(xy)] = 0$

$[x, T(x)](xy + yx) + x[xy + yx, T(x)] + x[x^2, T(y)] = 0$ (10)

Finally combining the last relation with (9), we get

$[x, T(x)](xy + yx) = 0$ for all $x, y \epsilon S$

$[x, T(x)](xy + y'x + yx + yx) = 0$

$[x, T(x)](xy + y'x) + 2[x, T(x)]yx = 0$

$[x, T(x)](xy + y'x)] = 0.$

Replacing y by $T(x)$ leads to $([x, T(x]\,)^2 = 0$. Since a semiprime semiring has no non zero central nilpotent elements. It follows immediately that $[x, T(x)] = 0$ for all $x \epsilon S$. Which is the assertion of the theorem.

Theorem 2.3: *Let T be a centralizing Right centralizer of a semiprime semiring S. Then T is a centralizer of S.*

Proof: Assumption follows that $T(xy) = xT(y)$ for all $x, y \epsilon S$. We intend to prove that $T(xy) = T(x)y$, since T is a centralizing Right Centralizer of S, BY theorem (2.2) leads us to T is commuting.

Thus $[x, T(x)] = 0$, $xT(x) + T'(x)x = 0$

Now linearizing the above by using

$x = x + y$ yields $xT(y) + yT(x) + T(x)y' + T(y)x' = 0$

Applying xy for x one obtains that

$xyT(y) + yT(xy) + T(xy)y' + T(y)xy' = 0$

$x[y,T(y)] + yxT(y) + T(y)xy' = 0$
$yxT(y) + T(y)xy' = 0$
$yT(xy) + T(y)xy' = 0$ (11)
Let $T(x) = x$, then $T(xy) = xy$.
Now $T(x)y + x'T(y) = xy + x'y = 0$.
Hence $T(x)y = xT(y)$ (12)
Let $z\epsilon S$ then $\big(T(xy) + x'T(y)\big)z = xyz + x'yz = 0$
Also $(T(xy) + x'T(y))z(T(xy) + x'T(y)) = 0$
Since S is semiprime, we have $\big(T(xy) + x'T(y)\big) = 0$.
From above, and using (12), we get $T(xy) = xT(y) = T(x)y$
Thus T is a centralizer.

Remark 2.4 : *Every centralizer is commuting, because $T(xx) = T(x)x = xT(x)$for all $x\ \epsilon S$. And hence is a centralizing Right centralizer.*

Remark 2.5 : *A mapping T of a semiprime semiring S is a centralizer if and only if it is a centralizing Right centralizer.*

Corollary 2.6 : *Let T be a commuting right centralizer of a semiprime semiring S, then*

$[x,y]T(x) = [T(x),y]x$ *holds for all $x, y\ \epsilon S$*

Proof: By assumption we have $T(xy) = xT(y)$. Since T is commuting, it follows immediately that $[x,T(x)] = 0$. Linearising this we arrive at
$[x,T(y)] + [y,T(x)] = 0$ (13)
Let us replace y by yx in (13) and using $[x,T(x)] = 0$ implies
$[x,T(yx)] + [yx,T(x)] = 0$
$[x,yT(x)] + [yx,T(x)] = 0$
$[x,y]T(x) + y[x,T(x)] + [y,T(x)]x + y[x,T(x)] = 0$
$[x,y]T(x) + [y,T(x)]x = 0$
$[x,y]T(x) + [T(x),y]x' = 0$
$[x,y]T(x) = [T(x),y]x$ for all $x, y\epsilon S$. The proof of the corollary is complete.

Remark 2.7 : *Suppose that S is a prime semiring and $T: S \to S$ is a commuting right centralizer. If $T(x)\ \epsilon\ Z(S)$ for all $x\epsilon S$, then $T = 0$ or S is comutative.*

Proof: Since $T(x)\ \epsilon Z(S)$, in this case we have $[T(x),y] = 0$ for all $y\epsilon S$

We also have $[x,y]\,T(x) = [T(x),y]x$ thus $[x,y]T(x) = 0$ (14)
Putting $y = yz$ in the above relation and using (14) yields

$([x,y]z + y[x,z])T(x) = 0$
$[x,y]zT(x) = 0$ for all $x,y,z\epsilon S$. Since S is prime $T(x) = 0$ or $[x,y] = 0$.
That is $T = 0$ or S is commutative.

Theorem 2.8 : *Let S be a semiprime semiring. Let T and H be right centralizers of S such that $xT(x) + H(x)x \epsilon Z(S)$ for all $x\epsilon S$. Then T and H are both centralizers.*

Proof:

By our assumption $xT(x) + H(x)x \epsilon Z(S)$ (15)
The linearized form of (15) is $xT(y) + yT(x) + H(x)y + H(y)x \epsilon Z(S)$ (16)
Thus $[xT(y) + yT(x) + H(x)y + H(y)x, x] = 0$. It is written in the form
$[xT(y) + yT(x) + H(y)x, x] = [H(x)y, x]'$ (17)
Using xy for y in (16) then apply (17) it leads to
$xT(xy) + xyT(x) + H(x)xy + H(xy)x$
$= (xxT(y) + xyT(x) + H(x)xy + xH(y)x) \ \epsilon Z(S)$
$(x(xT(y) + yT(x) + H(y)x) + H(x)xy \epsilon Z(S)$
Thus $[x(xT(y) + yT(x) + H(y)x) + H(x)xy, x] = 0$ for all $x,y \epsilon S$
$x[(xT(y) + yT(x) + H(y)x), x] + [H(x)xy, x] = 0$ (18)
$x[H(x)y, x]' + [H(x)xy, x] = 0$ (19)
That is $[xH(x)y, x]' + [H(x)xy, x] = 0$. Expanding and arranging the above result we arrive at $[H(x), x]y, x] = 0$, Hence $[H(x), x][y, x] + [[H(x), x], x]y = 0$ (20)
Now substituting yz for y in the last relation and again using (20) one obtains that
$[H(x), x][yz, x] + [[H(x), x], x]yz = 0$
$[[H(x), x], x]yz + [H(x), x]y[z, x] + [H(x), x][y, x]z = 0$
$[H(x), x]y[z, x] = 0$
$[x, H(x)]y[x, z] = 0$ (21)
In particular the relation (21) related to $[x, H(x)]y[x, H(x)] = 0$. By the semiprimeness of S, $[x, H(x)] = 0$. (22)
Thus the result follows H is commuting right centralizer. Hence theorem 2.3 is forced to conclude that H is a centralizer. Now we are able to prove T is commuting. We start with the hypothesis and the assumption, and also use relation (22) leads to
$[xT(x) + H(x)x, x] = 0$
$[xT(x), x] + [H(x)x, x] = 0$
$x[T(x), x] + [H(x), x]x = 0$
$x[x, T(x)]' + [x, H(x)]'x = 0$

$x[x,T(x)]' = [x,H(x)\,]x$

$x[x,T(x)]' = 0$

Replace x by x' the above relation becomes $x[x,T(x)\,] = 0$ (23)

Let us write (17) in the following form and using $[H(x),y] = [x,H(y)]$

we obtain $[xT(y) + yT(x),x] = [H(y)x,x]' + [H(x)y,x]'$

$x[T(y),x] + y[T(x),x] + [y,x]T(x) = [H(y),x]x' + H(x)[x,y]$

$x[T(y),x] + y[T(x),x] + [y,x]T(x) = [y,H(x)]x' + H(x)[x,y]$

$x[T(y),x] + y[T(x),x] + [y,x]T(x) = [H(x),y]x + H(x)[x,y]$ (24)

On the otherhand again by the hypothesis, we assume that

$[xT(x) + H(x)x,y] = 0.$

$[xT(x),y] + [H(x)x,y] = 0$

$[x,y]T(x) + x[T(x),y] + [H(x),y]x + H(x)[x,y] = 0$

$[y,x]'T(x) + x[T(x),y] = [H(x),y]x' + H'(x)[x,y]$ (25)

Adding (24) and (25) we arrive at

$x[T(y),x] + y[T(x),x] + x[T(x),y] = 0$ (26)

In particular, if $y = T(x)y$ in (26) and using (23) then applying (26) we get

$x[T(T(x)y),x] + T(x)y[T(x),x] + x[T(x),T(x)y] = 0$

$x[T(x)T(y),x] + T(x)y[T(x),x] + x[T(x),T(x)y] = 0$

$x(T(x)[T(y),x] + [T(x),x]T(y)) + T(x)y[T(x),x] + x[T(x),T(x)]y +$
$xT(x)[T(x),y] = 0$

$xT(x)[T(y),x] + T(x)y[T(x),x] + xT(x)[T(x),y] = 0$

$xT(x)[T(y),x] + T(x)(x'[T(y).x] + x'[T(x),y]) + xT(x)[T(x),y] = 0$

$[x,T(x)\,][T(y),x] + (xT(x) + T(x)x')[T(x),y] = 0$

$[x,T(x)\,][T(y),x] + [x,T(x)\,][T(x),y] = 0$ (27)

In (27) we introducing y by xy and using (26) we leads to the following.

$[x,T(x)\,][T(xy),x] + [x,T(x)\,][T(x),xy] = 0$

$[x,T(x)\,][xT(y),x] + [x,T(x)\,]([T(x),x]y + x[T(x),y]\,) = 0$

$[x,T(x)\,]x[T(y),x] + [x,T(x)\,][T(x),x]y + [x,T(x)\,]x[T(x),y] = 0$

$[x,T(x)\,](y'[T(x),x] + x'[T(x),y]\,) + [x,T(x)\,][T(x),x]y + [x,T(x)\,]x[T(x),y] = 0$

$[x,T(x)][T(x),x]y = [x,T(x)\,]y[T(x),x]$

$[x,T(x)\,][x,T(x)\,]'y = [x,T(x)\,]y[x,T(x)]'$

Replace $y\ by\ y'in$ the above , we get

$[x,T(x)\,][x,T(x)\,]y = [x,T(x)\,]y[x,T(x)]$ (28)

Putting y by yx in (28) and using (23) leads to

$[x,T(x)\,][x,T(x)\,]yx = [x,T(x)]yx[x,T(x)]$

$[x,T(x)\,][x,T(x)\,]yx = 0$ (29)

On the otherhand one can replace $y = y\,T(x)$ we get
$[x,T(x)\,][x,T(x)\,]yT(x)x = 0$
Replace y by y' the above relation becomes
$[x,T(x)\,][x,T(x)\,]y'T(x)x = 0$ (30)
Post multiplying (29) by $T(x)$ we obtain
$[x,T(x)\,][x,T(x)\,]yxT(x) = 0$ (31)
Now adding (31) and (30) and post multiplying by $[x,T(x)]$ one get that
$[x,T(x)\,][x,T(x)\,]y(xT(x) + T'(x)x) = 0$
$[x,T(x)\,][x,T(x)\,]y[x,T(x)\,] = 0$
$[x,T(x)\,][x,T(x)\,]y[x,T(x)\,][x,T(x)\,] = 0.$
BY the semiprimeness of S, $[x,T(x)\,][x,T(x)\,] = 0$ (32)
Finally substituting (32) in (28), immediately follows that $[x,T(x)\,] = 0$.
Thus T is commuting right centralizer. Hence theorem 2.3 forced to conclude that T is a centralizer.

Corollary 2.9 : *Let T and H be Right centralizers of a semiprime semiring S such that $xT(x) + H(x)x = 0$ for all $x\epsilon S$. Then both T and H are centralizers.*

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