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On the Structure of Some Groups Containing $PSL(2,27)$

Nouf M. Al-Raddadi

Department of Math, Umm Al-Qura University, Makkah, Saudi Arabia.
E-mail: noufmuh32@gmail.com

Abstract

In this paper, we will generate the wreath product of $PSL(2,27)$ by some groups. Also, we will generate the wreath product of $PSL(2,27)$ by symmetric and alternating groups. Also, we will construct the wreath product of $PSL(2,27)$ by some other wreath products.

Keywords: wreath product, $PSL(2,27)$.

1. INTRODUCTION

In 1998, Al-Amri,[2], has shown A_{kn+1} and S_{kn+1} can be generated symmetric generating and symmetrically generating using S_n and an element of order *k*. In 2001, Al-Amri,[3], has given symmetric and permutational generating set of the groups A_{kn+1} and S_{kn+1} using the wreath product

In 2009, Al-Shehri,[5], has used the Mathieu groups M_9, M_{10}, M_{12} to get S_{kn+1} and A_{kn+1}. Al-Amri and Al-Muhaimeed,[4], have studied different types of symmetric and permutational generating set of The projective.

The projective special linear groups $PSL(2,27)$ is group of non-singular 2 × 2 matrices over F_{27}. The $PSL(2,27)$ of order 9828 is one of the well know simple groups. It contains 16 conjugacy classes. It also contains four maximal subgroups of orders 351, 28, 26 and 12.

$PSL(2,27)$ can be generated using two permutations of orders 13 and 3 as follows;

$PSL(2,27) = \langle (3, 27, 25, 23, 21, 19, 17, 16, 14, 12, 10, 8, 6)(4,15, 13, 11, 9,7,5, 28, 26, 24, 22, 20, 18), (1, 2, 4)(5, 8, 24)(6, 21,10)(7, 16, 15)(9, 25, 28)(11, 13, 14)(12, 27,23)(17, 26, 18)(19,20, 22) \rangle.$

$PSL(2,27)$ can be finitely presented as follows[6];

$PSL(2,27) = \left\langle x, y, t : x^3 = y^{13} = t^2 = (yt)^2 = (xt)^3 = 1, y^{-3}xy^3 = xy^{-1}xy = y^{-1}xyx \right\rangle.$

We have found a new presentation of $PSL(2,27)$ as follows;

$PSL(2,27) = \left\langle x, y : x^{14} = y^{13} = (xy^{-1})^{13} = (x^2y^{-1}x^{-1}y^5x^{-1}y^{-1}) = (x^2y^2x^{-1}y^{-2}x^{-1}y^2x^2) = 1 \right\rangle.$

In this paper, we will show that we can generate the wreath product of $PSL(2,27)$ by some groups. Also, we will generate the wreath product of $PSL(2,27)$ by symmetric and alternating groups. Also, we will construct the wreath product of $PSL(2,27)$ by some other wreath products.

2. PRELIMINARY RESULTS

Definition 2.1. [1] Let A be a group of permutations of a finite set Ω_1 and B be a group of permutations of a finite set Ω_2. Assume that neither of Ω_1 nor Ω_2 is empty. The wreath product of A and B, denote by $A\ wr\ B$, is defined as $A\ wr\ B = A^{\Omega_2} \times_\theta B$ where A^{Ω_2} is the direct product of $|\Omega_2|$ copies of A and a mapping θ where $\theta : B \to Aut(A^{\Omega_2})$, is defined by $\theta_y(x) = x^y$, for all $x \in A^{\Omega_2}$ and $y \in B$. It follows that $|A\ wr\ B| = (|A|)^{|\Omega_2|} \times |B|$.

Theorem2.1.[1] $PSL(3,3)\ wr\ S_k$ could be generated using $PSL(3,3)\ wr\ C_k$ and an element of order 2, for all $k \geq 2$.

Theorem 2.3.[1] Let $1 \le a \ne b < n$ be any integers. Let n be an even integer and let G be the group generated by the (n-1)-cycle $(1, 2, ..., n-1)$ and the 3-cycle (n, a, b). Then $G = A_n$.

Theorem 2.4. [1] Let $1 < a < n$. Let $n = a\ m$ be any integer and let G be the group generated by the n-cycle $(1, 2, \ldots, n)$ and m-cycle(n, a, $2a$, ... ,$(m-1)a$).Then $G = C_m\ wr\ C_a$.

Theorem 2.5. [1] Let $1 < a < n$. Let G be the group generated by the n-cycle $(1, 2, \ldots, n)$ and 2- cycle (n, a). If $n = a\ m$, for some $m \in Z$ then $G \cong S_m wr\ C_a$.

Theorem 2.6. [1] Let $1 < a < n$. Let $n = a\ m$ be any integer and let G be the group generated by the n-cycle $(1, 2, \ldots, n)$, the permutation (1, 2)(a +1, a +2) (2a +1, 2a +2) … ((m -1) a +1, (m -1) a +2) and the 2-cycle (n,a). Then $G = S_m\ wr\ S_a$.

Theorem 2.7. [1] Let $1 < a \ne 2a < n$ be any integer . Let n be any integer . Let G be the group generated by the n-cycle $(1, 2, \ldots, n)$ and the 3-cycle (n, a, $2a$). If $n = a\ m$ and if m is an odd integer then $G = A_m\ wr\ C_a$.

3. THE RESULTS

Theorem 3.1. $PSL(2,27)\ wr\ C_k$ can be generated using two elements of order 14 and 13, for all integers k > 1.

Proof: Let σ=(1, 2, 3, … , 14k)(14k +1, 14k +2, … , 28k)
and τ = (k , 6k , 11k , 26k , 20k , 27k , 8k , 14k ,12k , 4k , 24k , 17k , 25k)(2k , 13k , 10k ,15k , 28k , 7k , 16k ,18k , 19k , 21k , 5k , 23k , 22k) be two permutations of order 14k and 13 respectively.

Let $G = \langle \sigma, \tau \rangle$. If k =1, then we get the trivial wreath product $PSL(2,27)$ *wr {id}*. Let k > 1 , conjugating τ by powers of σ^k we get (2k , 7k , 12k , 27k , 21k , 28k , 9k , k ,13k , 5k , 25k , 18k , 26k)(3k , 14k , 11k , 16k , 15k , 8k , 17k , 19k , 20k , 22k , 6k , 24k , 23k) , … , etc.

Let $\delta = \prod_{i=1}^{3} \tau^{\sigma^{ik}}$. It is not difficult to show that :

δ =(k , $4k$, $7k$, $10k$, $13k$, $2k$, $5k$, $8k$, $11k$, $14k$, $3k$, $6k$, $9k$, $12k$)($15k$, $18k$, $21k$, $24k$, $27k$, $16k$, $19k$, $22k$, $25k$, $28k$, $17k$, $20k$, $23k$, $26k$), which is the product two cycles each of order 14. Let $\alpha = \delta^5$ =(k , $2k$, $3k$, $4k$, $5k$, $6k$, $7k$, $8k$, $9k$, $10k$, $11k$, $12k$, $13k$, $14k$)($15k$, $16k$, $17k$, $18k$, $19k$, $20k$, $21k$, $22k$, $23k$, $24k$, $25k$, $26k$, $27k$, $28k$) .

Let $H = \langle \alpha, \tau \rangle$. Let $\theta : H \to PSL(2,27)$ be the mapping described before. It is not difficult to show that, θ is isomorphism and therefore

$H \cong PSL(2,27)$. Let $H_i = \left\langle \alpha^{\sigma^i}, \tau^{\sigma^i} \right\rangle \cong H$ for all $1 \le i \le k$. It is clear that each H_i acts on the set

$$\Gamma_i = \{k^{\sigma^i},\ (2k)^{\sigma^i},\ (3k)^{\sigma^i},\ \dots,\ (28k)^{\sigma^i}\}$$

for all i =1, 2, 3, … , k . Since $\bigcap_{i=1}^{k} \Gamma_i = \varphi$, then we get the direct product

$H_1 \times H_2 \times \dots \times H_k$,

where each $H_i \cong PSL(2,27)$. Let

$\beta = \alpha^{-1}\sigma$ =(1, 2, 3, … , k)(k +1, k +2, k +3, … ,$2k$)($2k$ +1, $2k$ +2,$2k$ +3, … , $3k$) … ($27k$ +1,$27k$ +2,$27k$ +3, … , $28k$),

which is the product of 28 cycles each of order k . Since β act on $H_1 \times H_2 \times \dots \times H_k$ by conjugation, then we get the wreath product $PSL(2,27)$ *wr* $C_k \subseteq G$. On the other hand, since

$\alpha\beta$ =(1, 2, 3, … , $14k$)($14k$ +1, $14k$ +2, … , $28k$) = σ

then $\sigma \in PSL(2,27)$ *wr* C_k . Therefore, $G = \langle \sigma, \tau \rangle \subseteq PSL(2,27)$ *wr* C_k .

Hence, $G = \langle \sigma, \tau \rangle \cong PSL(2,27)$ *wr* C_k . ◊

Corollary 3.2. $PSL(2,27)$ *wr*S_k can be generated using $PSL(2,27)$ *wr* C_k and an element of order 2, for all integers $k > 1$.

Proof: Let σ =(1, 2, 3, … , $14k$)($14k$ +1, $14k$ +2, … , $28k$) and

$\tau=(k, 6k, 11k, 26k, 20k, 27k, 8k, 14k, 12k, 4k, 24k, 17k, 25k)(2k, 13k, 10k, 15k, 28k, 7k, 16k, 18k, 19k, 21k, 5k, 23k, 22k)$ be two permutations of order $14k$ and 13 respectively. Let

$\mu=(1, k)(k+1, 2k)(2k+1, 3k)\ldots(27k+1, 28k)$.

By theorem 3.1, $\langle\sigma,\tau\rangle\cong PSL(2,27)\ wr\ C_k$ .Since, $\beta=(1,2,\ldots,k)(k+1,k+2,\ldots,2k)\ldots(27k+1,27k+2,\ldots,28k)\in PSL(2,27)\ wr\ C_k\cong\langle\sigma,\tau\rangle$,

then by theorem 2.1, $\langle\beta,\mu\rangle\cong S_k$.

Hence, it is clear that, $PSL(2,27)\ wr\ S_k\cong G=\langle\sigma,\tau,\mu\rangle$. ◊

Theorem 3.3. $PSL(2,27)\, wr A_k$ can be generated using $PSL(2,27)\, wr\ C_k$ and an element of order 3, for all odd integers $k\geq 3$.

Proof: Let $\sigma=(1, 2, 3, \ldots, 14k)(14k+1, 14k+2, \ldots, 28k)$

and $\tau=(k, 6k, 11k, 26k, 20k, 27k, 8k, 14k, 12k, 4k, 24k, 17k, 25k)(2k, 13k, 10k, 15k, 28k, 7k, 16k, 18k, 19k, 21k, 5k, 23k, 22k)$ be two permutations of order $14k$ and 13 respectively. Let

$\mu=(1, 2, 3)(k+1, k+2, k+3)(2k+1, 2k+2, 2k+3)\ldots(27k+1, 27k+2, 27k+3)$. By theorem 3.1, $\langle\sigma,\tau\rangle\cong PSL(2,27)\ wr\ C_k$.

Since, $\beta=(1, 2, \ldots, k)(k+1, k+2, \ldots, 2k)\ldots(27k+1, 27k+2, \ldots, 28k)\in PSL(2,27)\ wr\ C_k\cong\langle\sigma,\tau\rangle$, then by theorem 2.3, $\langle\beta,\mu\rangle\cong A_k$.

Hence, $G=\langle\sigma,\tau,\mu\rangle\cong PSL(2,27)\ wr\ A_k$. ◊

Theorem 3.4. Let $k=am$ be any integer with $1<a<k$. $PSL(2,27)\, wr\ (C_m\ wr\ C_a)$ can be generated using $PSL(2,27)\, wr\ C_k$ an element of order *m*.

Proof: Let $PSL(2,27)\, wr\ C_k$ be the group described in theorem 3.1. Let $\mu=(k, a, 2a, 3a, \ldots, (m-1)a)(2k, k+a, k+2a, \ldots, k+(m-1)a)(3k, 2k+a, 2k+2a, \ldots, 2k+(m-1)a)\ldots(28k, 27k+a, 27k+2a, \ldots, 27k+(m-1)a)$. Since $\beta=(1, 2, \ldots, k)(k+1, k+2, \ldots, 2k)\ldots(27k+1, 27k+2, \ldots, 28k)\in PSL(2,27)\ wr\ C_k$, then by theorem 2.4,

$\langle\beta,\mu\rangle\cong(C_m\ wr\ C_a)$.

Hence, $G=\langle PSL(2,27)\ wr\ C_k,\mu\rangle\cong PSL(2,27)\ wr(C_m\ \mathrm{wr}\ \mathrm{C}_a)$. ◊

Corollary3.5. $PSL(2,27)\,wr(S_m\ wr\ C_a)$ can be generated using $PSL(2,27)\,wr$ C_k and an element of order 2, where $k = am$ be any integer with $1 < a < k$.

Proof: Let $PSL(2,27)\,wr\ C_k$ be the group described in Theorem 3.1. Let μ=(k , a)(2k , k +a)(3k , 2k +a) … (28k , 27k +a). Since $\beta=$(1, 2, … , k)(k +1, k +2, … , 2k) … (27k +1,27k +2, … , 28k)$\in PSL(2,27)\ wr\ C_k$, then by theorem 2.5, $\langle \beta , \mu \rangle \cong (S_m\ wr\ C_a)$.

Hence, $G = \langle PSL(2,27)\ wr\ C_k , \mu \rangle \cong PSL(2,27)\ wr(\,S_m\ wr\ C_a)$. ◊

Corollary3.6. $PSL(2,27)\,wr(S_m\ wr\ S_a)$ can be generated using $PSL(2,27)\,wr$ C_k and two elements of order 2, where $k = am$ be any integer with $1 < a < k$.

Proof: Let $PSL(2,27)\,wr\ C_k$ be the group described in theorem 3.1. Let

μ =(k , a)(2k , k + a)(3k , 2k + a) … (28k , 27k +a) and

δ =(1, 2)(a +1, a +2)(2a +1, 2a +2) … ((m -1)a +1, (m -1)a +2)(k +1, k +2)(k +(a +1), k +(a +2))))(k +((2a +1), k +(2a +2)) … (k + ((m -1) a +1), k + ((m -1) a +2)(2k +1, 2k +2)(2k +(a +1), 2k +(a +2))(2k +(2a +1), 2k +(2a +2)) … (2k +((m -1) a +1), 2k + ((m -1) a +2)) … (27k +1, 27k +2)(27k +(a +1), 27k +(a +2))(27k +(2a +1), 27k +(2a +2))… (27k +((m -1) a +1), 27k +((m -1) a +2)).Since $\beta=$(1, 2, … , k)(k +1,k +2, … , 2k) … (27k +1,27k +2, … , 28k)$\in PSL(2,27)\ wr\ C_k$, then by theorem 2.6, $\langle \beta , \mu , \delta \rangle \cong (S_m wr\ S_a)$.

Hence,

$G = \langle PSL(2,27)\ wr\ C_k , \mu, \delta \rangle \cong PSL(2,27)\ wr(\,S_m\ wr\ S_a)$. ◊

Corollary3.7. $PSL(2,27)\,wr(A_m\ wr\ C_a)$ can be generated using $PSL(2,27)\,wr$ C_k and an element of order 3, where $k = am$ be any integer with $1 < a < k$ and $m > 5$ is an odd integer.

Proof: Let $PSL(2,27)\,wr\ C_k$ be the group described in theorem 3.1. Let

μ=(k , a , 2a)(2k , k +a , k +2a)(3k , 2k +a , 2k +2a) … (28k , 27k +a , 27k +2a). Since $\beta=$(1, 2, …, k)(k +1, k +2, … , 2k) … (27k +1,27k +2, … , 28k)$\in PSL(2,27)\ wr\ C_k$. Then by theorem 2.7,

$\langle \beta , \mu \rangle \cong (A_m wr\ C_a)$. Hence,

$G = \langle PSL(2,27)\ wr\ C_k, \mu \rangle \cong PSL(2,27)\ wr(A_m\ wr\ C_a)$. ◊

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