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Quadratic Form of Automorphism of a finite Abelian p-Group of Rank two

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Abstract

Formula for the number of automorphism of a finite abelian group of rank two is already determined. We can associate a quadratic from with finite Abelian group of rank two. We prove this quadratic form is positive definite.

Keywords: - Automorphism, Euler's function, number of divisor, quadratic form, positive definite matrix.

Mathematics Subject Classification: primary 20K27; secondary 15A63

1. INTRODUCTION

One of the famous problems in group theory is to find the number of automorphism of an abelian p-group. In [1], an explicit formula for the number of automorphism of a finite abelian p-group of rank two is indicated i.e., $Z_{p^m} \times Z_{p^n}$. The number f(m, n) distinct automorphism of group $Z_{p^m} \times Z_{p^n}$ with $m, n \in N \cup \{0\}$, can be associated with quadratic form. Our goal of our paper is to show that the quadratic form is positive definite

In [1], authors provides the formula for total number of automorphism of Group $Z_{p^m} \times Z_{p^n}$ are $\phi(p^m)\phi(p^n)(1+p)^{\left[\frac{\min\{m,n\}}{\max\{m,n\}}\right]} p^{2\min\{m,n\} - \left[\frac{\min\{m,n\}}{\max\{m,n\}}\right]}$

2. MAIN RESULTS

Let us consider the matrix $A_n = (a_{ij}) \in M_{n+1}(N)$ defined by $a_{ij} = f(i,j) \forall\, i,j = 0,1,2,\ldots,n$. Clearly, A_n is symmetric and so it induces quadratic form $\sum_{i,j=0}^{n} a_{ij} x^i y^j$. Now we compute the principal minors in the top left corner of A_n for this we have to find an explicit expression for $\det(A_k)$ for all $k = 0,1,2,\ldots,n$

Theorem 1:- For each $3 \leq k \leq n$, the following equality holds:-
$$\mathbf{det}(A_k) = (p-1)^{k-1}\left[p^{(k-1)^2 + \frac{k(k-1)}{2}}\right]\left[\frac{(p^3+p^2-p+1)^{k-2}[(p^2+p-1)^2-1] - (p-1)(p^3+p^2-p+1)^{k-3} + p}{(p^2+p-1)}\right]$$

Proof: - Let $k \in \{0,1,2,\ldots,n\}$ be fixed. Using [1], the determinant $\det(A_k)$ is given by:
$$\det\left(\phi(p^i)\phi(p^j)(1+p)^{\left[\frac{\min\{i,j\}}{\max\{i,j\}}\right]} p^{2\min\{i,j\} - \left[\frac{\min\{i,j\}}{\max\{i,j\}}\right]}\right)_{i,j=\overline{0,k}}$$

$$= \begin{vmatrix} 1 & \phi(p^1) & \phi(p^0)\phi(p^2) & \ldots & \phi(p^k) \\ \phi(p^1) & \phi(p^1)\phi(p^1)(1+p)^1 p^1 & \phi(p^1)\phi(p^2)p^2 & \ldots & \phi(p^1)\phi(p^k)p^2 \\ \phi(p^2) & \phi(p^2)\phi(p^1)p^2 & \phi(p^2)\phi(p^2)(1+p)^1 p^3 & \ldots & \phi(p^2)\phi(p^k)p^4 \\ \phi(p^3) & \phi(p^3)\phi(p^1)p^2 & \phi(p^3)\phi(p^2)p^4 & \ldots & \phi(p^3)\phi(p^k)p^6 \\ \phi(p^4) & \phi(p^4)\phi(p^1)p^2 & \phi(p^4)\phi(p^2)p^4 & \ldots & \phi(p^4)\phi(p^k)p^8 \\ \phi(p^5) & \phi(p^5)\phi(p^1)p^2 & \phi(p^5)\phi(p^2)p^4 & \ldots & \phi(p^5)\phi(p^k)p^{10} \\ \ldots. & \ldots & \ldots & \ldots & \ldots. \\ \phi(p^{k-1}) & \phi(p^{k-1})\phi(p^1)p^2 & \phi(p^{k-1})\phi(p^2)p^4 & \ldots & \phi(p^{k-1})\phi(p^k)p^{2k-2} \\ \phi(p^k) & \phi(p^k)\phi(p^1)p^2 & \phi(p^k)\phi(p^2)p^4 & \ldots & \phi(p^k)\phi(p^k)(1+p)^1 p^{2k-1} \end{vmatrix}$$

$$= \left[\prod_{i=0}^{k}\phi(p^i)\right]^2 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & p^2+p & p^2 & p^2 & p^2 & p^2 & \ldots & p^2 & p^2 \\ 1 & p^2 & p^4+p^3 & p^4 & p^4 & p^4 & \ldots & p^4 & p^4 \\ 1 & p^2 & p^4 & p^6+p^5 & p^6 & p^6 & \ldots & p^6 & p^6 \\ 1 & p^2 & p^4 & p^6 & p^8+p^7 & p^8 & \ldots & p^8 & p^8 \\ 1 & p^2 & p^4 & p^6 & p^8 & p^{10}+p^9 & \ldots & p^{10} & p^{10} \\ \ldots. & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots. \\ 1 & p^2 & p^4 & p^6 & p^8 & p^{10} & \ldots & p^{2k-2}+p^{2k-3} & p^{2k-2} \\ 1 & p^2 & p^4 & p^6 & p^8 & p^{10} & \ldots & p^{2k-2} & p^{2k}+p^{2k-1} \end{vmatrix}$$

We shall apply consecutive transformations $C_t = C_t - C_{t-1}$ for every $t = 1,2,3,..,k$, we get

$$= [\prod_{i=0}^{k} \Phi(p^i)]^2 \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & p^2+p-1 & -p & 0 & 0 & \dots & 0 \\ 1 & p^2-1 & p^2(p^2+p-1) & -p^3 & 0 & \dots & 0 \\ 1 & p^2-1 & p^2(p^2-1) & p^4(p^2+p-1) & -p^5 & \dots & 0 \\ 1 & p^2-1 & p^2(p^2-1) & p^4(p^2-1) & p^6(p^2+p-1) & \dots & 0 \\ 1 & p^2-1 & p^2(p^2-1) & p^4(p^2-1) & p^6(p^2-1) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & p^2-1 & p^2(p^2-1) & p^4(p^2-1) & p^6(p^2-1) & \dots & -p^{2k-3} \\ 1 & p^2-1 & p^2(p^2-1) & p^4(p^2-1) & p^6(p^2-1) & \dots & p^{2k-2}(p^2+p-1) \end{vmatrix}$$

$$= [\prod_{i=0}^{k} \Phi(p^i)]^2 p^{(k-1)^2} \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & p^2+p-1 & -1 & 0 & 0 & \dots & 0 \\ 1 & p^2-1 & p(p^2+p-1) & -1 & 0 & \dots & 0 \\ 1 & p^2-1 & p(p^2-1) & p(p^2+p-1) & -1 & \dots & 0 \\ 1 & p^2-1 & p(p^2-1) & p(p^2-1) & p(p^2+p-1) & \dots & 0 \\ 1 & p^2-1 & p(p^2-1) & p(p^2-1) & p(p^2-1) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & p^2-1 & p(p^2-1) & p(p^2-1) & p(p^2-1) & \dots & -1 \\ 1 & p^2-1 & p(p^2-1) & p(p^2-1) & p(p^2-1) & \dots & p(p^2+p-1) \end{vmatrix}$$

We shall apply consecutive transformations $R_t = R_t - R_{t-1}$ for every $t = 1,2,3,..,k$, we get

$$= [\prod_{i=0}^{k} \Phi(p^i)]^2 p^{(k-1)^2} \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & p^2+p-1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -p & (p^3+p^2-p+1) & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -p^2 & (p^3+p^2-p+1) & -1 & \dots & 0 & 0 \\ 0 & 0 & 0 & -p^2 & (p^3+p^2-p+1) & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & -p^2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & (p^3+p^2-p+1) & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & -p^2 & (p^3+p^2-p+1) \end{vmatrix}$$

We shall apply transformations on the matrix C_r in order to put it into lower triangular matrix form.

So, we consider following consecutive transformations

$C_2 = C_2 + \frac{1}{p^2+p-1} C_1$, $C_3 = C_3 + \frac{(p^2+p-1)}{(p^3+p^2-p+1)(p^2+p-1)-p} C_2$,

$C_4 = C_4 + \frac{(p^3+p^2-p+1)(p^2+p-1)-p}{(p^3+p^2-p+1)^2(p^2+p-1)-p(p^3+p^2-p+1)-p^2} C_3$,

$C_5 = C_5 + \frac{(p^3+p^2-p+1)^2(p^2+p-1)-p(p^3+p^2-p+1)-p^2}{(p^3+p^2-p+1)^3(p^2+p-1)-p(p^3+p^2-p+1)^2-p^2[(p^3+p^2-p+1)+1]} C_4$, and

$C_r = C_r + \frac{(p^3+p^2-p+1)^{r-3}(p^2+p-1)-p(p^3+p^2-p+1)^{r-4}-p^2[(p^3+p^2-p+1)^{r-5}+..+(p^3+p^2-p+1)+1]}{(p^3+p^2-p+1)^{r-2}(p^2+p-1)-p(p^3+p^2-p+1)^{r-3}-p^2[(p^3+p^2-p+1)^{r-4}+..+(p^3+p^2-p+1)+1]} C_{r-1} \; \forall r = 6,7,\ldots,k$

we get lower triangular matrix whose diagonal elements are as follows :-

$a_{00} = 1, a_{11} = p^2+p-1, a_{22} = \frac{(p^3+p^2-p+1)(p^2+p-1)-p}{(p^2+p-1)}$,

$a_{33} = \frac{(p^3+p^2-p+1)^2(p^2+p-1)-p(p^3+p^2-p+1)-p^2}{(p^3+p^2-p+1)(p^2+p-1)-p}$,

$a_{44} = \frac{(p^3+p^2-p+1)^3(p^2+p-1)-p(p^3+p^2-p+1)^2-p^2[(p^3+p^2-p+1)+1]}{(p^3+p^2-p+1)^2(p^2+p-1)-p(p^3+p^2-p+1)-p^2}$, *and*

$a_{rr} = \frac{(p^3+p^2-p+1)^{r-2}(p^2+p-1)-p(p^3+p^2-p+1)^{r-3}-p^2[(p^3+p^2-p+1)^{r-4}+..+(p^3+p^2-p+1)+1]}{(p^3+p^2-p+1)^{r-3}(p^2+p-1)-p(p^3+p^2-p+1)^{r-4}-p^2[(p^3+p^2-p+1)^{r-5}+..+(p^3+p^2-p+1)+1]} \; \forall r = 5,6,\ldots k$

Hence

$\det(A_k)=$

$[\prod_{i=0}^{k} \Phi(p^i)]^2 [p^{(k-1)^2}][(p^3+p^2-p+1)^{k-2}(p^2+p-1) - p(p^3+p^2-p+1)^{k-3} - p^2[(p^3+p^2-p+1)^{k-4}+..+(p^3+p^2-p+1)+1]$

]

$\det(A_k)=$

$[\prod_{i=0}^{k} \Phi(p^i)]^2 [p^{(k-1)^2}][(p^3+p^2-p+1)^{k-2}(p^2+p-1) - p(p^3+p^2-p+1)^{k-3} - p^2[\frac{(p^3+p^2-p+1)^{k-3}-1}{(p^3+p^2-p+1)-1}]$

]

$\det(A_k)=$

$[\prod_{i=0}^{k} \Phi(p^i)]^2 [p^{(k-1)^2}][\frac{(p^3+p^2-p+1)^{k-2}(p^2+p-1)^2-p(p^2+p-1)(p^3+p^2-p+1)^{k-3}-p(p^3+p^2-p+1)^{k-3}+p}{(p^2+p-1)}]$

$\det(A_k)=(p-1)^{k-1}\left[p^{(k-1)^2+\frac{k(k-1)}{2}}\right][\frac{(p^3+p^2-p+1)^{k-2}[(p^2+p-1)^2-1]-(p-1)(p^3+p^2-p+1)^{k-3}+p}{(p^2+p-1)}]$

Now, the following two corollaries are obvious from theorem 1.

Corollary 1:- The quadratic form $\sum_{i,j=0}^{k} f(i,j)x^i y^j$ induced by the matrix A_k is positive definite, for all $k \in \overline{0,n}$.

Proof:-Let $k \in \{0,1,2,\ldots,n\}$ be fixed. Using [1], the determinant $\det(A_k)$ is given by: $\det\left(\Phi(p^i)\Phi(p^j)(1+p)^{\left[\frac{\min\{i,j\}}{\max\{i,j\}}\right]} p^{2\min\{i,j\}-\left[\frac{\min\{i,j\}}{\max\{i,j\}}\right]}\right)_{i,j=\overline{0,k}}$

Case 1:- k=0, then $\det(A_1) = 1 > 0$

Case 2:- k=1, then $\det(A_k) = \begin{vmatrix} 1 & \phi(p^1) \\ \phi(p^1) & \phi(p^1)\phi(p^1)(1+p)^1 p^1 \end{vmatrix} = (p-1)^2(p^2+p-1)$

We know that $(p^2+p-1) > 0$ for every prime, hence $\det(A_k) > 0$

Case 3:- k=2, then $\det(A_k) = p^2(p-1)^4[p^2(p^2+p-1)^2 - (p^2-1)^2]$
Here $p^2(p^2+p-1)^2 > (p^2+p-1)^2 > (p^2-1)^2$
Hence, $\det(A_k) > 0$

Case 4:- k>2, we have $(p^2+p-1)^2 > p$ => $(p^2+p-1)^2 > p$, hence $(p^2+p-1)^2 - 1 > p-1$
Therefore, $\det(A_k) > 0$
Hence, quadratic form $\sum_{i,j=0}^{k} f(i,j)x^i y^j$ induced by the matrix A_k is positive definite, for all $k \in \overline{0,n}$

Corollary 2:- For each $k \in N$, all eigenvalues of the matrix A_k are positive.

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