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A study on the Classes of Semirings and Ordered Semirings

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Abstract

In this paper we have proved that if S is an E–inverse semiring and (S,) is a rectangular band, then we get two different outcomes under two different conditions.

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1. INTRODUCTION

The concept of semirings was first introduced by Vandiver in 1934. Semirings arise naturally in such diverse areas of mathematics as combinatorics, functional analysis, topology, graph theory, Euclidean geometry, probability theory, commutative, non-commutative ring theory and the mathematical modeling of quantum physics and parallel computation systems. . The developments of semirings and ordered semirings in this direction require semigroup techniques. It is well known that if the multiplicative structure of an ordered semiring is a rectangular band, then its additive structure is a band. In the recent papers on ordered semirings, the works of M.Satyanarayana [8,9,10] has studied how far the properties of multiplicative structure are reflected in the additive structure and vice-versa.

In this paper we have two sections. Section one contains classes of E-inverse semirings and section two deals with classes of Ordered left regular semirings.

2. PRELIMINARIES:

Definition 2.1:

A semiring is called a E –inverse semiring if ab + acb = ab for every a, b, c in S.

Definition 2.2:

A semigroup (S, •) is weak commutative if abc = bac for all a, b, c in S.

Definition 2.3:
A semigroup (S, •) is rectangular band if a = aba for all a, b in S.

Definition 2.4:
In a semiring S, an element ***a*** is Multiplicatively Subidempotent if $a + a^2 = a$. A semiring S is Multiplicatively Subidempotent if and only if each of its elements is Multiplicatively Subidempotent.

Definition 2.5:
A semigroup (S, •) is left (right) singular if a + b = a (a + b = b) for all a, b in S.

Definition 2.6:
A semigroup (S, •) is a band if $a^2 = a$ for all a in S.

Definition 2.7:
An element x in a t.o.s.r is minimum (maximum) if $x \leq a$ ($x \geq a$) for every a in S.

Definition 2.8:
A semigroup (S, +) is semilattice if (S, +) is a band and commutative.
i.e a + a = a and a + b = b + a for all a, b in S.

Definition 2.9:
In a totally ordered semiring (S, +, •, ≤)
(i) (S, +, ≤) is positively totally ordered (p.t.o), if $a + b \geq a, b$ for all a, b in S.
(ii) (S, •, ≤) is positively totally ordered (p.t.o), if $ab \geq a, b$ for all a, b in S.
(iii) (S, +, ≤) is negatively totally ordered (n.t.o), if $a + b \leq a, b$ for all a, b in S.
(iv) (S, •, ≤) is positively totally ordered (n.t.o), if $ab \leq a, b$ for all a, b in S.

3. CLASSES OF E – INVERSE SEMIRINGS:

The concept of an E-inversive semigroup was constructed by G.Thierrin [13] and developed by Lallement and Petrich. But Petrich [6, 7] was studied in some what different form of Lallement and Petrich. This type of semigroups recently reapeared in papers Hall and Munn, F.Catino and M.M.Miccoli, Margolis and Pin and H.Mitsch.

Theorem 3.1: Let S be an E – inverse semiring and (S, ·) be a rectangular band.
(i) If (S, ·) is right cancellative, then (S, ·) is multiplicatively subidempotent.
(ii) If (S, ·) is weak commutative, then $a^2 + a = a^2$ for all a in S.

Proof: **(i)** Given that (S, ·) is rectangular band then aba = a for all a, b in S
From the definition of E-inverse semiring ab + acb = ab------ (1)
$\Rightarrow abc + acbc = abc \Rightarrow abc + ac = abc \Rightarrow ab + a = ab \Rightarrow aba + a^2 = aba$
$\Rightarrow a + a^2 = a$
Hence (S, ·) is multiplicatively subidempotent

(ii) By hypothesis S is an E – inverse semiring then ab + acb = ab -------(i)
Since (S, ·) is weak commutative then $ab + abc = ab \Rightarrow bab + babc = bab$
By using the definition of rectangular band in above it takes the form b + bc = b
$\Rightarrow b^2 + bcb = b^2 \Rightarrow b^2 + b = b^2$ -------(ii)

Now let us check that $a^2 + a = a^2$

For this we take the equation (i) as $ab + cab = ab \Rightarrow aba + caba = aba \Rightarrow a + ca = a$

$\Rightarrow a^2 + aca = a^2 \Rightarrow a^2 + a = a^2$ --------(iii)

Now let us take, $a + ca = a \Rightarrow ac + cac = ac \Rightarrow ac + c = ac \Rightarrow cac + c^2 = cac$

$\Rightarrow c + c^2 = c$

Since $c^3 = c$ we have $c^2 + c^3 = c^2 \Rightarrow c^2 + c = c^2$ --------------(iv)

Thus we obtained $a^2 + a = a^2$ for all a, b, c in S

Theorem 3.2 : Suppose S is an E –inverse semiring. If ‘1’ is multiplicative and also additive identity. Then

- **(i)** $ab + ab = ab$.
- **(ii)** $ac = a$ and $cb = b$.

Proof: **(i)** By the definition of E –inverse semiring $ab + acb = ab$ for all a, b in S ---- (I)

It implies $a [b + cb] = ab \Rightarrow a [1 + c] b = ab$

Since 1 is multiplicative identity and also additive identity then $1 + c = c$

Thus we obtain $acb = ab$ Now equation (1) can take the form as $ab + ab = ab$

(ii) Let us check whether $ac = a$

For this let us consider $b =1$ in equation (I) we get $a.1 + ac.1 = a.1$

$\Rightarrow a.1 + ac = a \Rightarrow a (1 + c) = a \Rightarrow ac = a$

Now we proceed on to show that $cb = b$

For this let us take $a = 1$in first condition it leads to $1.b + 1.cb = 1.b$

$\Rightarrow (1 + 1.c) b = b \Rightarrow cb = b$

Hence $ac = a$ and $cb = b$

Theorem 3.3 : If S is an E –inverse semiring and $(S, \cdot)$ is left singular semigroup, then $(S, +)$ is a band.

Proof : Consider $(S, \cdot)$ is left singular which implies that $cb = c$ and $ab = a$

Suppose if S is an E –inverse semiring then $ab + acb = ab$

$\Rightarrow a (b + cb) = ab \Rightarrow a (b + c) = ab \Rightarrow ab + ac = ab \Rightarrow a + a = a$ for all a in S

Hence $(S, +)$ is band

CLASSES OF TOTALLY ORDERED LEFT REGULAR SEMIRINGS:

This section contains some results on totally ordered left regular semirings using the properties like p.t.o, n.t.o, etc. A left regular semiring is a semiring in which $a + ab + b = a$ for all a, b in S.

Theorem 4.1: Let S be a commutative idempotent semiring. Define a relation $\leq$ on S such that $a \leq b$ if and only if S is a left regular semiring. If ‘e’ is multiplicative identity which is also an additive identity then S is a totally ordered semiring and e is a maximum element of S.

Proof: Let us define a relation $\leq$ on S such that $a \leq b$ if and only if S is a left regular semiring and also ‘e’ is multiplicative identity which is also an additive identity

Since S is an idempotent semiring we have $a + a = a$ and $a^2 = a$ for all ‘a’ in S

Assume S is a left regular semiring then $a + ab + b = a$ for all a in S

Now we have to check that whether $(S, +, \cdot, \leq)$ is a totally ordered semiring
We have $a + a.a + a = a + a + a = a + a = a$
Thus $a \leq a$ implies '$\leq$' is reflexive
Let $a \leq b$ and $b \leq a \Rightarrow a + ab + b = a$ and $b + ba + a = b$
Again consider $a = a + ab + b = a [e + b] + b = ab = a [b + e] = ab + a$
Since $(S, \cdot)$ is commutative
$a = ba + a = b [e + a] + a = b + ba + a = b$
$\Rightarrow$ '$\leq$' is anti symmetric

Let $a \leq b$ and $b \leq c \Rightarrow a + ab + b = a$ and $b + bc + c = b$
To prove $a \leq c$ for this we have to prove that $a + ac + c = a$
Consider $a = a + ab + b = a + a [b + bc + c] + b + bc + c$
$= a + ab + abc + ac + b [e + c] + c = a + ab [e + c] + ac + bc + c$
$= a + abc + ac + bc + c = a + a [b + e]c + bc + c = a + a[e + b]c + bc + c$
$= a + ac + abc + bc + c = a + [a + ab + b] c + c = a + ac + c$
i.e $a \leq c$ which implies '$\leq$' is transitive

Next we prove that compatibility with respect to multiplication
For this we have to prove that $a \leq b$ implies $a + ab + b = a$ then for any c in S $ac \leq bc$
Now $a \leq b$ implies $a + ab + b = a \Rightarrow (a + ab + b)c = ac \Rightarrow ac + abc + bc = ac$
Since S is idempotent semiring implies $ac + abcc + bc = ac \Rightarrow ac + acbc + bc = ac$
Thus $ac \leq bc$
Similarly we can prove that $ca \leq cb$
Next we prove that compatibility with respect to addition
For this we have to prove that $a \leq b$ implies $a + ab + b = a$ then for any c in S
$a + c \leq b + c$
Now $a \leq b$ implies $a + ab + b = a \Rightarrow (a + ab + b) + c = a + c$
Also we know that if S is a left regular semiring with multiplicative and additive identity 'e' then $(S, +)$ is left singular thus $a + b = a$ for all a, b in S
$(a + c) + (a + c) (b + c) + b + c = a + c$
Thus $a + c \leq b + c$
Similarly we can prove that $c + a \leq c + b$
Therefore $(S, +, \cdot, \leq))$ is a totally ordered semiring
Now $a + a.e + e = a + a + e = a + a = a \Leftrightarrow a \leq e$ for all a in S
Hence e be the maximum element

Theorem 4.2: Let S be a totally ordered left regular and $(S, +)$ is semilattice. If $(S, +)$ is p.t.o (n.t.o), then $(S, \cdot)$ is n.t.o (p.t.o).

Proof: Given that S is left regular then $a + ab + b = a$, for all a, b in S
Since $(S, +)$ is semilattice then $(S, +)$ is additive idempotent and commutative this implies $a + a (b + b) + b = a \Rightarrow a + ab + ab + b = a \Rightarrow ab + a + ab + b = a$
$\Rightarrow ab + a = a$ - - - (A)
$\Rightarrow a = ab + a \geq ab \Rightarrow a \geq ab$
Suppose $ab > b \Rightarrow ab + a \geq b + a \Rightarrow a \geq b + a \Rightarrow a \geq a + b \Rightarrow a + b \leq a$
Which contradicts the hypothesis that $(S, +)$ is p.t.o $\Rightarrow ab \leq b$
Therefore $ab \leq a$ and $ab \leq b$
Hence $(S, \cdot)$ is n.t.o

Similarly we can prove that (S, ·) is p.t.o if (S, +) is n.t.o.

Theorem 4.3: Let S be a totally ordered left regular semiring. If (S, +) is p.t.o (n.t.o.) which contains multiplicative identity 1, then 1 is minimum (maximum) element.

Proof: Given that S is a totally ordered left regular semiring then

$a + a.1 + 1 = a$, for all a, 1 in S implies $a + a + 1 = a$

Suppose (S, +) is p.t.o then we have $a + 1 \geq 1 \Rightarrow a + a + 1 \geq a + 1$

$\Rightarrow a \geq a + 1 \geq 1 \Rightarrow a \geq 1$

Thus 1 is the minimum element.

Also if (S, +) is n.t.o i.e., $a + 1 \leq 1$ which implies $a + a + 1 \leq a + 1$

$\Rightarrow a \leq a + 1 \leq 1 \Rightarrow a \leq 1$

Therefore 1 is the maximum element

REFERENCES

[1] V. R. Daddi and Y.S. Pawar, "On Completely Regular Ternary Semiring", Novi Sad J. Math. ol. 42, No. 2, 2012, 1-7.

[2] Jonathan S.Golan, "Semirings and Affine Equations over Them: Theory and Applications", Kluwer Academic publishers.

[3] Jonathan S.Golan, " Semirings and their Applications" , Kluwer Academic Publishers.

[4] M.P.Grillet :"Semirings with a completely semisimple additive semigroups". Jour. Aust. Math. Soc., Vol.20 (1975), 257-267.

[5] N.Kehayopulu, "On a characterization of regular duo le Semi groups", Maths. Balkonica 7 (1977), 181- 186.

[6] M. Petrich, "Introduction to semigroups" Merill", Columbus, Ohio (1973).

[7] M. Petrich, "Archimedean classes in an ordered Semigroup IV". Czechoslovak Mathematical Jour. 37(112) 1987, praha, 86 -119.

[8] M.Satyanarayana, "Naturally totally ordered semigroups" Pacific Journal of Mathematics, Vol.77, No. 1, 1978.

[9] M.Satyanarayana , "Positively ordered semigroups". Lecture notes in Pure and Applied Mathematics, Marcel Dekker, Inc., Vol.42 (1979).

[10] M.Satyanarayana, "On the additive semigroup of ordered semirings", Semigroup forum vol.31 (1985), 193- 199.

[11] M. K.Sen, S. K. Maity and K. P. Shum, "Clifford Semirings And Generalized Clifford Semirings" Taiwanese Journal of Mathematics Vol. 9, No. 3, pp. 433-444, September 2005.

[12] M.K. Sen, S.K.Maity, "Semirings Embedded in a Completely Regular Semiring", Acta Univ. Palacki. Olomuc., Fac.rer. nat., Mathematica 43 (2004) 141–146.

[13] G. Theirrin, "The Sytantic monoid of a hypercode", Semigroup form 6 (1973), 227 - 231.