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A note on the composition of p-Eisenstein polynomials

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Abstract

We give a simple counter example to point out an error in Theorem 2.4 of [*Journal of the Chungcheong Mathematical Society*, **22**, No. 3 (2009), 497--506] and give a necessary condition for the result to be true.

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Introduction and the main theorem.
Eisenstein Irreducibility Criterion states that if $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial with integer coefficients and p is a prime such that $p \nmid a_n$, $p|a_i$, $0 \leq i \leq n-1$, $p^2 \nmid a_0$, then $f(x)$ is irreducible over the field Q of rational numbers ([2, Theorem 3.10.2]). A polynomial belonging to $Z[x]$ satisfying Eisenstein Irreducibility Criterion with respect to the prime p is referred to as p-Eisenstein.
For example, $f(x) = x + 2$, $g(x) = x^2 + 4x + 2$ are 2-Eisenstein polynomials but the composition $fog(x) = f\big(g(x)\big) = (x^2 + 4x + 2) + 2 = (x + 2)^2$ is not 2-Eisenstein. So we need to add a necessary condition in Theorem 2.4 of [1] to make it valid.

Theorem. *If $f(x)$, $g(x)$ are p-Eisenstein polynomials and degree of $f(x) \geq 2$, then $fog(x)$ is a p-Eisenstein polynomial.*

Proof. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in Z[x]$ be a p-Eisenstein polynomial of degree $n \geq 2$ and $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m \in Z[x]$ be another p-Eisenstein polynomial. Then the composition
$fog(x) = f\big(g(x)\big) = a_0 + a_1g(x) + a_2(g(x))^2 + \cdots + a_n(g(x))^n$
is a polynomial of degree mn with leading coefficient $a_nb_m^n$ which is not divisible by p. Since p divides the coefficient of x^i for each i, $0 \leq i \leq mn - 1$, we need only check that $p^2 \nmid f\big(g(0)\big)$. Now $f\big(g(0)\big) = f(b_0) = a_0 + a_1b_0 + \cdots + a_nb_0^n = a_0 + c$, where $c = a_1b_0 + \cdots + a_nb_0^n$. In view of the fact that $f(x)$ is p-Eisenstein and $n \geq 2$, it follows that $p|a_0$, $p^2|c$ and hence $p|f(b_0)$. If $p^2|f(b_0)$, then as $p^2|c$, we have $p^2|(f(b_0) - c)$, which implies that $p^2|a_0$. A contradiction. Hence $f\big(g(x)\big)$ is p-Eisenstein and is irreducible over Q.

Example. Let $p \geq 3$ be a prime number and $\varphi_p(x) = x^{p-1} + x^{p-2} + \cdots + 1$. Then $\varphi_1(x) = \varphi_p(x + 1)$ is a p-Eisenstein polynomial. It follows from the above theorem that $\varphi_1 o\ \varphi_1(x)$ is a p-Eisenstein polynomial and hence irreducible over Q.

References

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