Journal of Advances in Algebra (AA).
ISSN 0973-6964 Volume 9, Number 1 (2016), pp. 1–8

http://www.ripublication.com/aa.htm

Ideals Of The Ring Of Higher Dimensional Dual Numbers

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Abstract

Let R be a commutative ring with unity and let $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ be the ring of the higher dimensional dual numbers ring. The purpose of this article is to characterize the ideal structure of the ring $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ and its relation to the ideal structure of the ring R. In particular, a describtion of the maximal ideals, prime ideals, finitely generated ideals and pure ideals are also given.

AMS subject classification: 13A15, 13A25.
Keywords: Polynomial ring, Dual numbers ring, Maximal ideal, Prime ideal, Principal ideal ring, PM-ring.

1. Introduction

Throghout this article R is a commutative ring with a nonzero identity 1. Let $R[x_1, x_2, \ldots, x_n]$ be the polynomial ring over R with indetemenants $x_1, x_2, \ldots, x_n$. Then the factor ring of $R[x_1, x_2, \ldots, x_n]$ modulo the ideal $\langle x_i x_j \rangle_{1\le i,j}^{n}$, which is the ideal of $R[x_1, x_2, \ldots, x_n]$ generated by the set $\{x_i x_j\}_{1\le i,j}^{n}$, has the form

$$R[x_1, x_2, \ldots, x_n]/\left\langle x_i x_j \right\rangle_{1\le i,j}^{n}$$

Suppose the set $R[\alpha_1, \alpha_2, \ldots, \alpha_n] = \{a_0 + a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n : a_i \in R$ and $\alpha_i\alpha_j = 0$ for $1 \le i, j \le n\}$. Then $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ over the operations, addition pointwise and multiplication defined by $(a_0 + a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n)(b_0 + b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n) = a_0b_0 + (a_0b_1 + a_1b_0)\alpha_1 + (a_0b_2 + a_2b_0)\alpha_2 + \ldots + (a_0b_n + a_nb_0)\alpha_n$, forms a commutative ring with unity 1. Furthermore, there is a ring isomorphism ϕ between $R[x_1, x_2, \ldots, x_n]/\left\langle x_i x_j \right\rangle_{1\le i,j}^{n}$ and $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ which is given by $\phi(a_0 + a_1x_1 + a_2x_2 + \ldots + a_nx_n + \left\langle x_i x_j \right\rangle_{1\le i,j}^{n}) = a_0 + a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$. As a special case, If $n = 1$, then the ring $R[\alpha_1] \cong R[x_1]/\langle x_1^2 \rangle$ is precisely the ring of dual numbers. Hence, Vasantha et.al in [7] have named $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ as the higher dimensional dual numbers ring.

There are connections between some of the results in this paper and the earlier work of Asfahani et.al. [5], where some results in this paper are somewhat similar to those in [5], but there are some key differences since we use different rings.

Throughout the paper, $U(R)$ denotes the group of units of R , and $Idem(R)$, $Nil(R)$ denote the set of idemptents of R and the set of nilpotents of R respectively. Any undefined notation or terminology is standard, as in [1]. For more about the ring $R[x_1, x_2, \ldots, x_n]$ consult [6].

2. Ideal Structure of the ring $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$

In this section, we shall describe the ideals in the ring $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ in terms of those in R.

We start this section with the following lemma which is necessary to construct the ideals of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$.

Lemma 2.1. Let R be a ring and n a positive integer. Then $J = \{a_0 + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n : a_i \in I_i$, where I_i ideals of R with $I_0 \subseteq I_i$ for $1 \leq i \leq n\}$ is an ideal of R. Moreover, we denote such ideal J as $J = I_0 + I_1\alpha_1 + I_2\alpha_2 + \cdots + I_n\alpha_n$.

The next lemma shows that if I is an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$, then we can construct a family of ideals of R depending on I.

Lemma 2.2. Let R be a ring, and I is an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. Then $I_i = \{r \in R :$ there are $r_0, r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n \in R$ with $r_0 + r_1\alpha_1 + \cdots + r_{i-1}\alpha_{i-1} + r\alpha_i + r_{i+1}\alpha_{i+1} + \cdots , r_n\alpha_n \in I\}$ is an ideal of R for $0 \leq i \leq n$. Moreover, $I_0 \subseteq I_i$ for $1 \leq i \leq n$.

Proof. Let $a \in I_i$, and $r \in R$. So there are $a_0, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in R$ with $x = a_0 + a_1\alpha_1 + \cdots + a_{i-1}\alpha_{i-1} + a\alpha_i + a_{i+1}\alpha_{i+1} + \cdots + a_n\alpha_n \in I$. Since I is an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$, $rx = ra_0 + ra_1\alpha_1 + \cdots + ra_{i-1}\alpha_{i-1} + ra\alpha_i + ra_{i+1}\alpha_{i+1} + \cdots + ra_n\alpha_n \in I$. Hence, $ra \in I_i$. Now, let $b \in I_0$. So there are $a_1, a_2, \ldots, a_n \in R$ with $y = b + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in I$. Then since I is an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$, $y\alpha_i = (b + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)\alpha_i = b\alpha_i \in I$. Thus, $b \in I_i$, and $I_0 \subseteq I_i$, for $1 \leq i \leq n$. ∎

Now we use Lemma 2.1 and Lemma 2.2 to describe the ideals of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$.

Theorem 2.3. Let R be a ring. Then each ideal I of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ has the form $I = I \cap J$, where $J = I_0 + I_1\alpha_1 + I_2\alpha_2 + \cdots + I_n\alpha_n$, where I_i is an ideal of R defined as in Lemma 2.2 for $0 \leq i \leq n$.

Proof. Let I be an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. Then by Lemma 2.1 and Lemma 2.2,

$J = I_0 + I_1\alpha_1 + I_2\alpha_2, \cdots + I_n\alpha_n$ is an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. It is also clear that $I \subseteq J$. Therefore, $I = I \cap J$. ■

Corollary 2.4. Let R be a ring. Then any ideal $I = I \cap J$ of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ where $J = I_0 + I_1\alpha_1 + I_2\alpha_2 + \cdots + I_n\alpha_n$ is uniquely determined by $I_0, I_1, I_2, \ldots, I_n$.

The following example shows that, for an ideal I of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$, it is unnecessary that $J = I_0 + I_1\alpha_1 + I_2\alpha_2 + \cdots + I_n\alpha_n \subseteq I$. That means, an ideal I of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$, need not have the form $I = I_0 + I_1\alpha_1, I_2\alpha_2, \ldots, I_n\alpha_n$.

Example 2.5. Let $I = \langle \alpha_1 + 2\alpha_2 \rangle$ be an ideal of $\mathbb{Z}[\alpha_1, \alpha_2]$ generated by $\alpha_1 + 2\alpha_2$.It is clear that $I_0 = 0$, $I_1 = \mathbb{Z}$, $I_2 = 2\mathbb{Z}$. Hence $I \subseteq \mathbb{Z}\alpha_1 + 2\mathbb{Z}\alpha_2$. The other inclusion is not correct since $\alpha_1 \in \mathbb{Z}\alpha_1 + 2\mathbb{Z}\alpha_2$ and $\alpha_1 \notin I$ otherwise $\alpha_1 = (\alpha_1 + 2\alpha_2)(a + b\alpha_1 + c\alpha_2) = a\alpha_1 + 2a\alpha_2$. Hence $a = 1$, and $2a = 0$, which is a contradiction.

Also the following example shows that if $I = I \cap (I_0 + I_1\alpha_1 + I_2\alpha_2 + \cdots + I_n\alpha_n)$ is an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$, then it is not true in general that $I_0 \subseteq I$.

Example 2.6. Let $I = \langle 2 + \alpha_1 \rangle$ be an ideal of $\mathbb{Z}[\alpha_1]$ generated by $2 + \alpha_1$. Then it is easy to show that I has the form $I = \langle 2 + \alpha_1 \rangle \cap (2\mathbb{Z} + \mathbb{Z}\alpha_1)$. Suppose that $2\mathbb{Z} \subseteq \langle 2 + \alpha_1 \rangle$, then $2 = (2 + \alpha_1)(a + b\alpha_1)$ for some $a + b\alpha_1 \in \mathbb{Z}[\alpha_1]$. Thus we have $2a = 2$ and $2b + a = 0$. Solving these two equations produces $a = 1$ and $2b = -1$ which is a contradiction. Hence $2\mathbb{Z} \nsubseteq \langle 2 + \alpha_1 \rangle$.

By adding some suitable assumptions on the ideals of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. one can prove the next result easily.

Lemma 2.7. Let R be a ring, and let $I = I \cap (I_0 + I_1\alpha_1 + I_2\alpha_2 + \cdots + I_n\alpha_n)$ be an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. If $\alpha_i \in I$, for all $1 \le i \le n$, then

1. $I_0 \subseteq I$, and
2. $I = I_0 + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n$.

3. Prime and Maximal ideals of the ring $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$

In this section, we will describe the prime and maximal ideals of the ring $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. We also prove some elementary results concerning the Nilradical and the Jacobson radical of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$.

Lemma 3.1. Let R be a ring, and let I_0 be an ideal of R. Then

$$R[\alpha_1, \alpha_2, \ldots, \alpha_n]/(I_0 + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n) \cong R/I_0$$

Proof. Define the mapping

$$\theta : R[\alpha_1, \alpha_2, \ldots, \alpha_n] \longrightarrow R/I_0$$

As $\theta(a_0 + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) = a_0 + I_0$. It is clear that θ is an onto ring homomorphism with kernel, $\ker(\theta) = I_0 + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n$. Then the result is obtained by applying the first isomorphism theorem for rings. ■

Theorem 3.2. Let R be a ring. Then

1. P is prime ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$, if and only if $P = P_0 + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n$ for some prime ideal P_0 of R.

2. M is maximal ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$, if and only if $M = M_0 + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n$ for some maximal ideal M_0 of R.

Proof.

(1) Suppose that P is prime ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. By Theorem 2.3, $P = P \cap (P_0 + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n)$. Now since $\alpha_i^2 = 0 \in P$, for $1 \leq i \leq n$, we have $\alpha_i \in P$. Thus by Lemma 2.7, $P = P_0 + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n$. The rest of the proof obtained by applying Lemma 3.1, and using the fact R/I is an integral domain if and only if I is a prime ideal.

(2) Similar to part (1), but we use the fact that R/I is a field if and only if I is a maximal ideal. ■

In commutative ring theory, the Jacobson radical (Nilradical) of the ring R is the intersection of all maximal (prime) ideals of R (see [1]).

Corollary 3.3. Let R be a ring. Then

1. $J(R[\alpha_1, \alpha_2, \ldots, \alpha_n]) = J(R) + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n$

2. $Nil(R[\alpha_1, \alpha_2, \ldots, \alpha_n]) = Nil(R) + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n$.

Proof.

1.

$$\begin{aligned} J(R[\alpha_1, \alpha_2, \ldots, \alpha_n]) &= \text{Intersection of all maximal ideals of } R[\alpha_1, \alpha_2, \ldots, \alpha_n] \\ &= \bigcap_{M \in Max(R)} (M + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n) \\ &= \Big(\bigcap_{M \in Max(R)} M\Big) + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n \\ &= J(R) + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n \end{aligned}$$

2.

$$\begin{aligned} Nil(R[\alpha_1,\alpha_2,\dots,\alpha_n]) &= \text{Intersection of all prime ideals of } R[\alpha_1,\alpha_2,\dots,\alpha_n] \\ &= \bigcap_{P\in Spec(R)} (P + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n) \\ &= (\bigcap_{P\in Spec(R)} P\ ) + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n \\ &= Nil(R) + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n \end{aligned}$$

■

Corollary 3.4. Let R be a ring. Then R is local ring if and only if $R[\alpha_1,\alpha_2,\dots,\alpha_n]$ is so.

Recall that a ring R is called a PM-ring if for each prime ideal in R is contained in a unique maximal ideal, for more see [2].

Theorem 3.5. Let R be a ring. Then R is PM-ring if and only if $R[\alpha_1,\alpha_2,\dots,\alpha_n]$ is PM-ring.

Proof. Suppose that R is PM-ring and let P be a prime ideal of $R[\alpha_1,\alpha_2,\dots,\alpha_n]$ then by Theorem 3.2, $P = P_0+R\alpha_1+R\alpha_2+\cdots+R\alpha_n$ for some P_0 a prime ideal of R. Since R is Gelfand ring, P_0 is contained in a unique maximal ideal, say M_0 Also, again by Theorem 3.2, $M = M_0 + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n$ is maximal ideal of $R[\alpha_1,\alpha_2,\dots,\alpha_n]$. Note that, P is contained in M . Now, If there exists a maximal ideal N of $R[\alpha_1,\alpha_2,\dots,\alpha_n]$ that contains P, then by Theorem 3.2, $N = N_0 + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n$ for some maximal ideal N_0 of R such that

$$P_0 + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n = P \subseteq N = N_0 + R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n$$

It is clear that P_0 is contained in N_0 which is a contradiction except if $N_0 = M_0$. Hence, $N_0 = M_0$.Therefore, $R[\alpha_1,\alpha_2,\dots,\alpha_n]$ is PM-ring. By the same argument we can prove the converse direction. ■

4. Finitely Generated Ideals of $R[\alpha_1,\alpha_2,\dots,\alpha_n]$

In this section, we will study and describe the finitely generated ideals of the ring $R[\alpha_1,\alpha_2,\dots,\alpha_n]$. Then we will characterize when $R[\alpha_1,\alpha_2,\dots,\alpha_n]$ is a principal ideal ring.

Theorem 4.1. Let R be a ring, and let $a_0+a_1\alpha_1+a_2\alpha_2+\cdots+a_n\alpha_n \in R[\alpha_1,\alpha_2,\dots,\alpha_n]$. Then the ideal generated by $a_0 + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ has the form

$$I = \langle a_0 + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n\rangle = I \cap (I_0 + I_1\alpha_1 + I_2\alpha_2 + \cdots + I_n\alpha_n),$$

where $I_0 = \langle a_0\rangle$, and $I_i = \langle a_0, a_i\rangle$ for $1 \le i \le n$.

Proof. Suppose that I is an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ generated by $a_0 + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$. Then by Theorem 2.3, $I = I \cap (I_0 + I_1\alpha_1 + I_2\alpha_2 + \cdots + I_n\alpha_n)$. Now let $x \in I_0$. So there exist $b_1, b_2, \ldots, b_n \in R$ such that $(x + b_1\alpha_1 + \cdots + b_n\alpha_n) = (a_0 + a_1\alpha_1 + \cdots + a_n\alpha_n)(c_0 + c_1\alpha_1 + \cdots + c_n\alpha_n)$ for some $c_0 + c_1\alpha_1 + \cdots + c_n\alpha_n \in R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. Thus, $x = a_0c_0$ and hence $I_0 \subseteq \langle a_0 \rangle$. Also since $a_0 + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in I$ we have $\langle a_0 \rangle \subseteq I_0$. To prove that $I_i = \langle a_0, a_i \rangle$ for $1 \leq i \leq n$. Let $x \in I_i$. Then there exist $b_0, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n \in R$ such that $(b_0 + b_1\alpha_1 + \cdots + x\alpha_i + \ldots + b_n\alpha_n) = (a_0 + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)(c_0 + c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n)$ for some $c_0 + c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n \in R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. Hence we have $x = a_0c_i + a_ic_0 \in \langle a_0, a_i \rangle$. That is $I_i \subseteq \langle a_0, a_i \rangle$. To prove the other inclusion, let $x \in \langle a_0, a_i \rangle$. So $x = a_0t + a_is$ for some $s, t \in R$. Thus, there exist $a_0s, a_1s, a_2s, \ldots, a_{i-1}s, a_{i+1}s, \ldots, a_ns \in R$ with $a_0s + a_1s\alpha_1 + a_2s\alpha_2 + \cdots + x\alpha_i + \cdots + a_ns\, \alpha_n \in I$. Therefore, $\langle a_0, a_i \rangle \subseteq I_i$. ■

Corollary 4.2. Let R be a ring. Then R is principal ideal ring if and only if $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ is so.

Proof. Let I be an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. Then by Theorem 2.3, $I = I \cap (I_0 + I_1\alpha_1 + I_2\alpha_2 + \cdots + I_n\alpha_n)$. Since R is principal ideal ring, $I_i = \langle a_i \rangle$ for $1 \leq i \leq n$. Also since $I_0 \subseteq I_i$ for $1 \leq i \leq n$ we have $b \in R$ such that a_i divides b for $1 \leq i \leq n$. Thus,

$$\begin{aligned} I &= I \cap (\langle b \rangle + \langle a_1 \rangle \alpha_1 + \langle a_2 \rangle \alpha_2 + \cdots + \langle a_n \rangle \alpha_n) \\ &= I \cap (\langle b \rangle + \langle a_1, b \rangle \alpha_1 + \langle a_2, b \rangle \alpha_2 + \cdots + \langle a_n, b \rangle \alpha_n) \\ &= \langle b + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \rangle \end{aligned}$$

Therefore, $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ is a principal ideal ring. The other direction is obvious. ■

The following corollary describes the form of the finitely generated ideal of the ring $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$.

Corollary 4.3. Let R be a ring, and let $x_j = \left\{ \sum_{i=0}^{n} a_{ij}\alpha_i \right\}$ be a family of elements of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. Then the ideal generated by the finite set $\{x_j\}_{j=1}^{m}$ has the form

$$I = \langle x_1, x_2, \ldots x_m \rangle = I \cap (I_0 + I_1\alpha_1 + I_2\alpha_2 + \cdots + I_n\alpha_n),$$

where $I_0 = \left\langle \{a_{0j}\}_{j=1}^{m} \right\rangle$, and $I_i = \left\langle \{a_{0j}, a_{ij}\}_{j=1}^{m} \right\rangle$ for $1 \leq i \leq n$.

5. The Pure ideals of the ring $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$

In this section, we will exhibit the relationship between the pure ideals of R and the pure ideals of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. Recalling that an deal I of R is said to be pure if for every element $x \in I$, there exits $y \in I$ such that $xy = x$.

Lemma 5.1. Let R be a ring, and let $I = I \cap (I_0 + I_1\alpha_1 + I_2\alpha_2 + \cdots + I_n\alpha_n)$ be an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. If $I_0 = I_1 = I_2 = \cdots = I_n$, then

1. $I_0 \subseteq I$. and
2. $I = I_0 + I_0\alpha_1 + I_0\alpha_2 + \cdots + I_0\alpha_n$.

Proof. It is clear that $I \subseteq (I_0 + I_0\alpha_1 + I_0\alpha_2 + \cdots + I_0\alpha_n)$. Now let $a_0 + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in (I_0 + I_0\alpha_1 + I_0\alpha_2 + \cdots + I_0\alpha_n)$. Hence by the assumption $a_i \in I_0$ for all $0 \leq i \leq n$. So there exist $b_{1,i}, b_{2,i}, \ldots, b_{n,i} \in R$ such that $a_i + b_{1,i}\alpha_1 + b_{2,i}\alpha_2 + \cdots + b_{n,i}\alpha_n \in I$. Since I is an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$, we have $a_i\alpha_i = (a_i + b_{1,i}\alpha_1 + b_{2,i}\alpha_2 + \cdots + b_{n,i}\alpha_n)\alpha_i \in I$ for all $1 \leq i \leq n$. Using the same argument and since $I \subseteq (I_0 + I_0\alpha_1 + I_0\alpha_2 + \cdots + I_0\alpha_n)$, we deduce $b_{t,0}\alpha_t \in I$ for all $1 \leq t \leq n$. Thus, $a_0 \in I$. Therefore, $a_0 + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in I$, and $(I_0 + I_0\alpha_1 + I_0\alpha_2 + \cdots + I_0\alpha_n) \subseteq I$. This proves (2). ■

Lemma 5.2. Let R be a ring, and let $I = I \cap (I_0 + I_1\alpha_1 + I_2\alpha_2 + \cdots + I_n\alpha_n)$ be an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. If I is a pure ideal, then

1. $I_0 = I_i$ for all $1 \leq i \leq n$.
2. I_0 is a pure ideal of R.

Proof. Suppose that $I = I \cap (I_0 + I_1\alpha_1 + I_2\alpha_2 + \cdots + I_n\alpha_n)$ is an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. Now

1. By the costruction of the ideal I in Theorem 2.3, we have $I_0 \subseteq I_i$ for all $\leq i \leq n$. Hence, it is enough to prove the other inclusion. Let $a \in I_k$ for any $1 \leq k \leq n$. So there are $b_0, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n \in R$ such that $b_0 + b_1\alpha_1 + \cdots + a\alpha_k + \cdots + b_n\alpha_n \in I$. Since I is a pure ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$, there exists $c_0 + c_1\alpha_1 + \cdots + c_k\alpha_k + \cdots + c_n\alpha_n \in I$ such that $b_0 + b_1\alpha_1 + \cdots + a\alpha_k + \cdots + b_n\alpha_n = (b_0 + b_1\alpha_1 + \cdots + a\alpha_k + \cdots + b_n\alpha_n)(c_0 + c_1\alpha_1 + \cdots + c_k\alpha_k + \cdots + c_n\alpha_n)$. That is, $a = b_0c_k + ac_0 \in I_0$. Hence, $I_0 = I_k$.
2. Let $x \in I_0$. So there exist $y_1, \cdots, y_n \in R$ such that $x + y_1\alpha_1 + \cdots + y_n\alpha_n \in I$. Hence, there is $z_0 + z_1\alpha_1 + \cdots + z_n\alpha_n \in I$ such that $x + y_1\alpha_1 + \cdots + y_n\alpha_n = (x + y_1\alpha_1 + \cdots + y_n\alpha_n)(z_0 + z_1\alpha_1 + \cdots + z_n\alpha_n)$, So $x = xz_0$ and this ends the proof since $z_0 \in I$. ■

De Marco [3] has studied purity and projectivity of ideals. Among many results he has proved that is if A is an ideal of the ring R, and if $\{a_1, a_2, \ldots, a_n\}$ is a finite subset of A, then there exists $b \in A$ such that $a_i = a_ib$ for all $1 \leq i \leq n$. This result will be used in the following lemma.

Lemma 5.3. Let R be a ring, and let $I = I_0 + I_o\alpha_1 + I_o\alpha_2 + \cdots + I_0\alpha_n$ be an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. If I_0 is a pure ideal of R, then I is pure ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$.

Proof. Let I_0 be a pure ideal of R. If $a_0 + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in I$, then exists $b \in I_0 \subseteq I$ such that $a_i = a_i b$ for all $0 \leq i \leq n$. Hence, $a_0 + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = (a_0 + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)b$ which completes the proof. ■

Gathering the results in Lemma 5.1, Lemma 5.2 and Lemma 5.3 produces the following theorem.

Theorem 5.4. Let R be a ring, and let I be an ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$. Then I is pure ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ if and only if $I = I_0 + I_0\alpha_1 + I_0\alpha_2 + \cdots + I_0\alpha_n$ with I_0 a pure ideal of R.

The following result is a direct consequence of Theorem 3.2 and Theorem 5.4.

Corollary 5.5. Let R be a ring, and let I be a pure ideal of $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$.Then I never be maximal nor prime ideal.

References

[1] Atiyah MF, MacDonald IG. Introduction to Commutative Algebra. Boston, MA, USA: Addison-Wesley Publishing Company, 1969.

[2] G. De Marco, A. Orsatti, Commutative rings in which every prime ideal is contained in a unique maximal ideal, Proc. Amer. Math. Soc., 1971, 30 (3): 459–466.

[3] G. De Marco. Projectivity of pure ideals, Rend. Sem. Mat. Univ. Padova,1983, 68: 289–304.

[4] Kai Long, Qichuan Wang, Lianggui Feng. Morphic property of a quotient ring over polynomial ring. Bulletin of the Korean Mathematical Society 2013, 50: 1433–1439.

[5] Nasr-Isfahani A., Moussavi, A. On a quotient of polynomial rings, Commun Algebra 2010; 38: 567–575.

[6] Thomas Hungerford, Algebra, Graduate Texts in Mathematics, vol. 73, Springer-Verlag, New York-Berlin, 1980.

[7] Vasantha Kandasamy W.B., Smarandache F., Dual Numbers, Zip publishing, Ohio,2012.